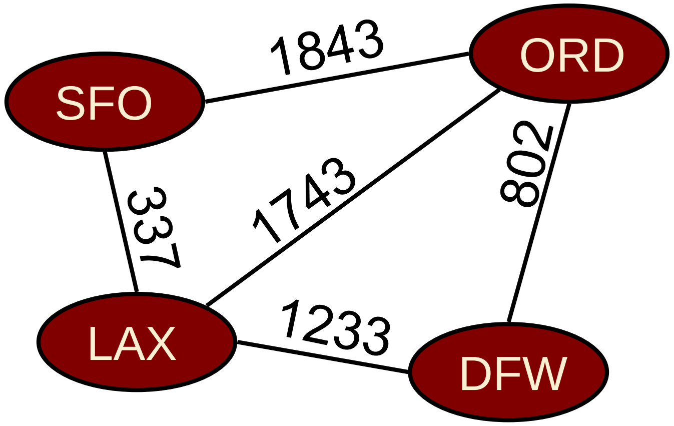
Graphs – Breadth First Search



Outline

- BFS Algorithm
- BFS Application: Shortest Path on an unweighted graph
- Unweighted Shortest Path: Proof of Correctness

Outline

- **BFS Algorithm**
- BFS Application: Shortest Path on an unweighted graph
- Unweighted Shortest Path: Proof of Correctness

Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with $|\boldsymbol{V}|$ vertices and $|\boldsymbol{E}|$ edges takes $\boldsymbol{O}(|\boldsymbol{V}|+|\boldsymbol{E}|)$ time
- BFS can be further extended to solve other graph problems
  - Cycle detection
  - **Find and report a path with the minimum number of edges between two given vertices**

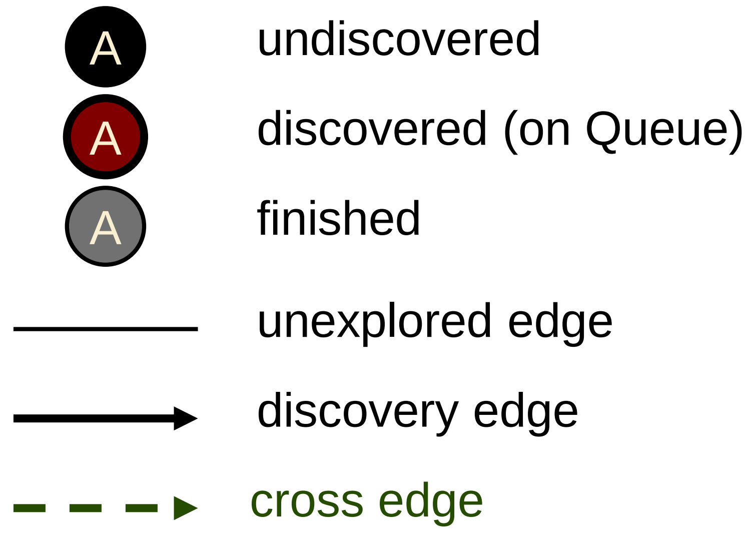
BFS Algorithm Pattern

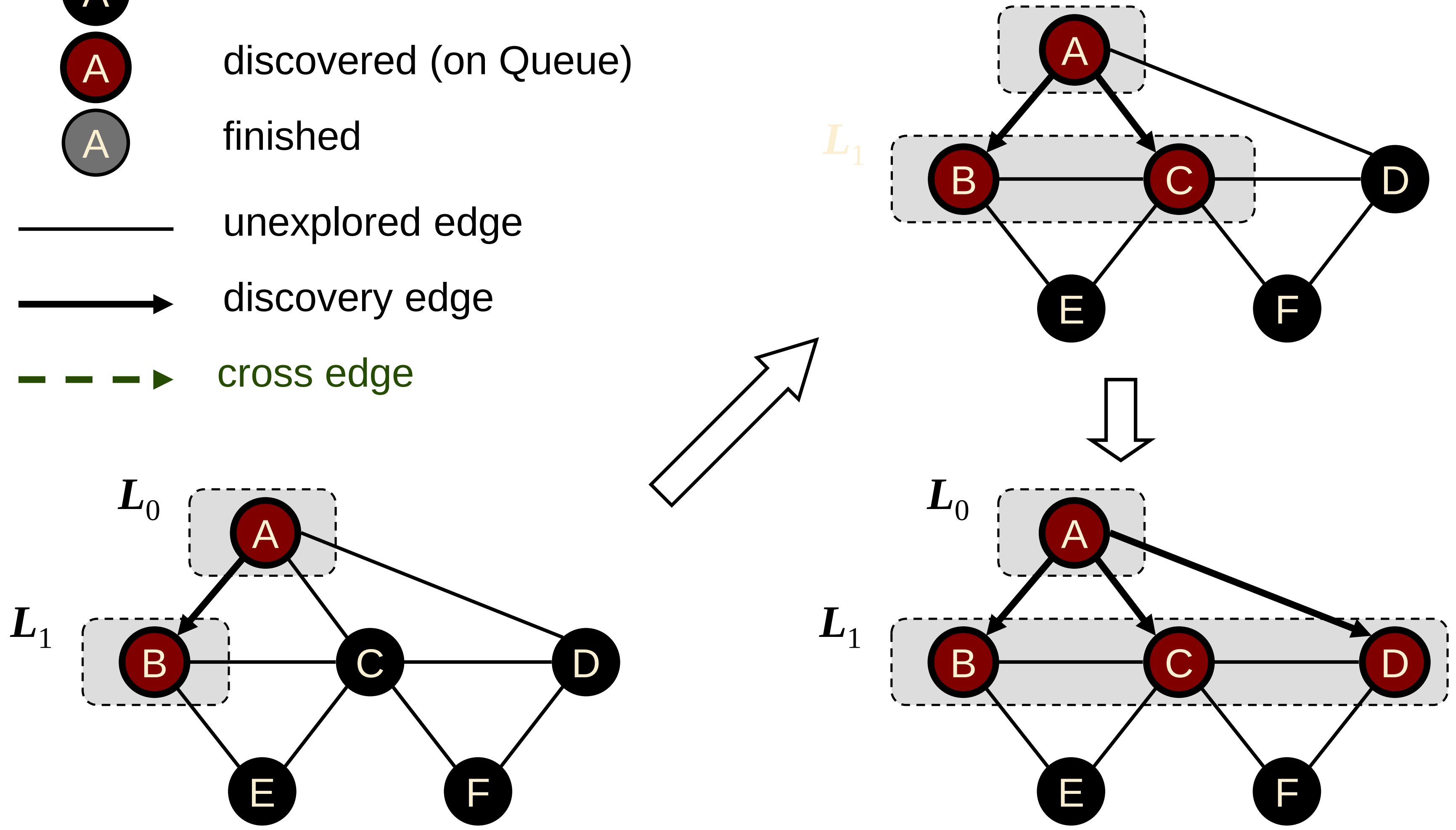
```
BFS(G,s)
Precondition: G is a graph, s is a vertex in G
Postcondition: all vertices in G reachable from s have been visited
    for each vertex u Î V[G]
        color[u] ¬ BLACK //initialize vertex
    colour[s] ¬ RED
    Q.enqueue(s)
    while Q ¹ Æ
        u ¬ Q.dequeue()
        for each v Î Adj[u] //explore edge (u,v)
            if color[v] = BLACK
                colour[v] ¬ RED
                Q.enqueue(v)
        colour[u] ¬ GRAY
```

BFS is a Level-Order Traversal

- Notice that in BFS exploration takes place on a wavefront consisting of nodes that are all the same distance from the source *s*.
- We can label these successive wavefronts by their distance: L_0, L_1, …

BFS Example


A
undiscovered
A
discovered (on Queue)
A
finished
unexplored edge
discovery edge
cross edge


L_0
L_1
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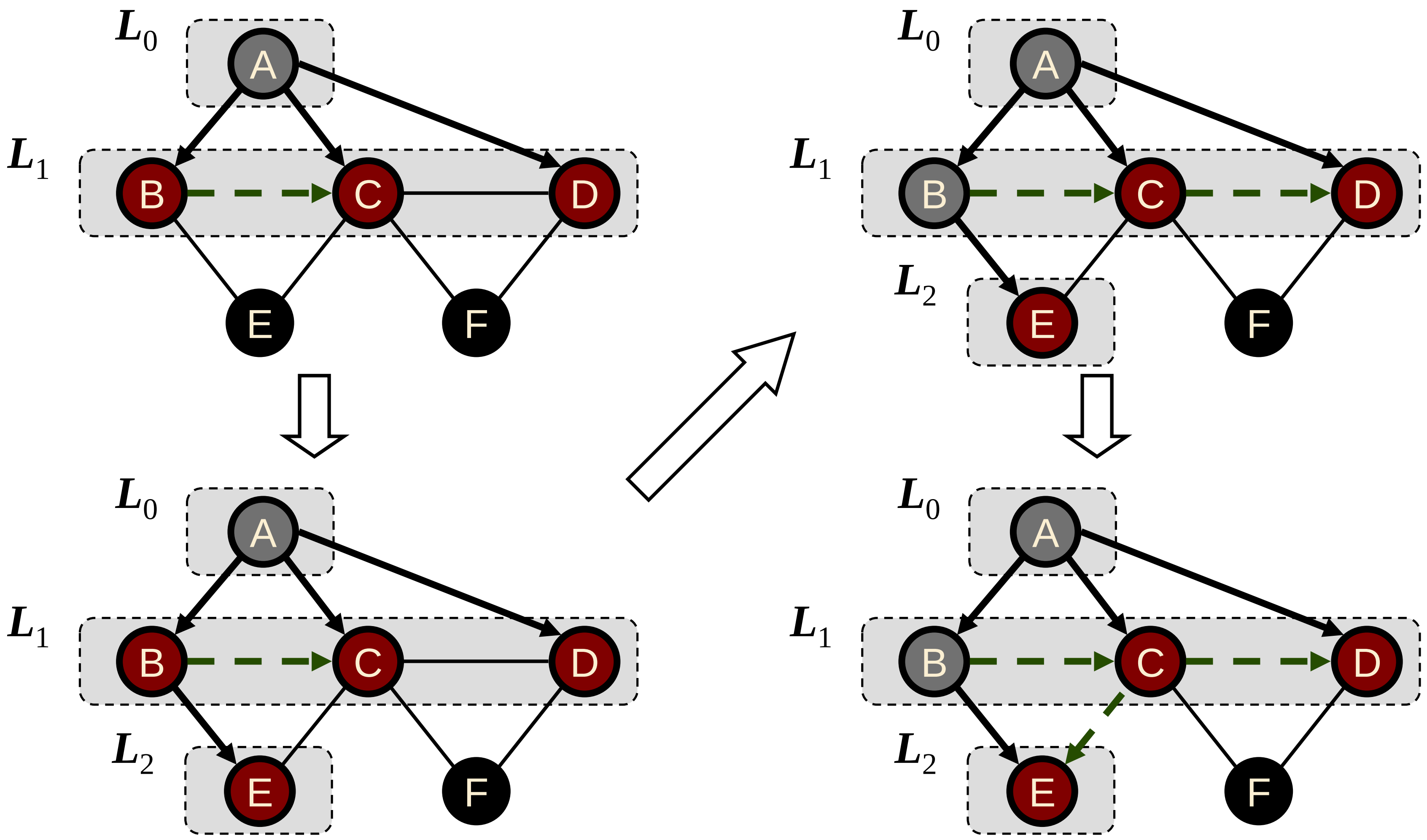
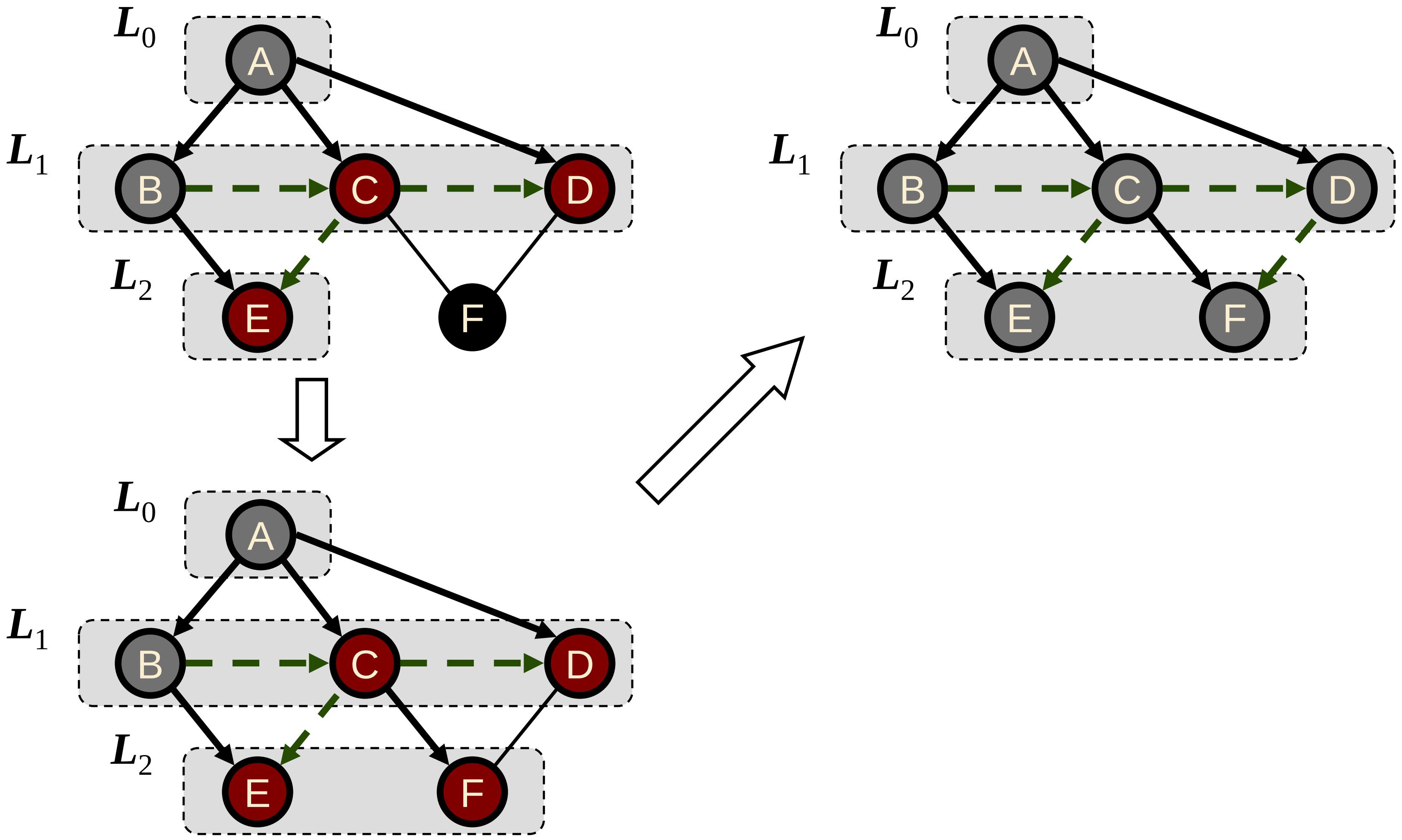
B
C
D
E
F
L_1
A
B
C
D
E
F
L_0
L_1
A
B
C
D
E
F

BFS Example (cont.)



BFS Example (cont.)



Properties

Notation

G_s: connected component of s

Property 1

$BFS(G, s)$ visits all the vertices and edges of G_s

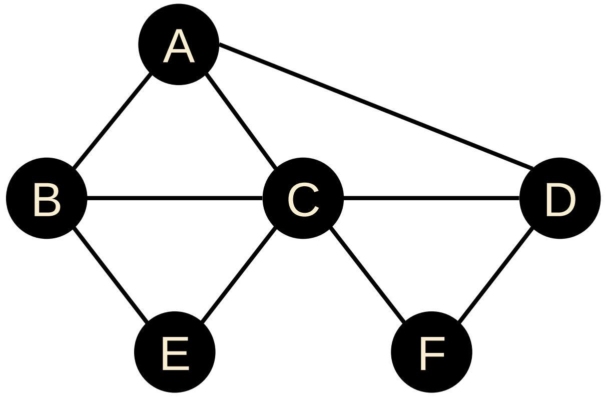
Property 2

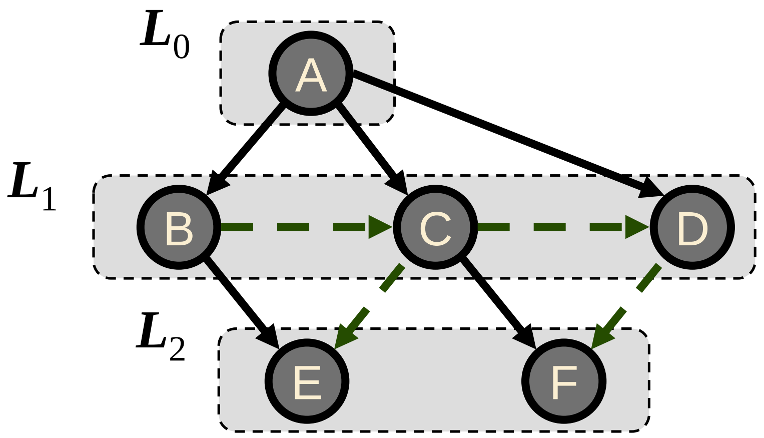
The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of G_s

Property 3

For each vertex v in L_i

- ❑ The path of T_s from s to v has i edges
- ❑ Every path from s to v in G_s has at least i edges





Analysis

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled three times
  - once as BLACK (undiscovered)
  - once as RED (discovered, on queue)
  - once as GRAY (finished)
- Each edge is considered twice (for an undirected graph)
- Each vertex is placed on the queue once
- Thus BFS runs in $O(|V|+|E|)$ time provided the graph is represented by an adjacency list structure

Applications

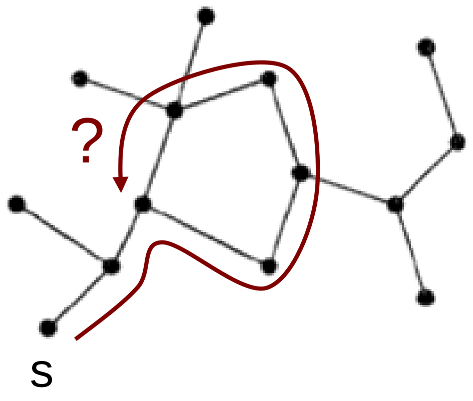
- BFS traversal can be specialized to solve the following problems in $O(|V|+|E|)$ time:
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

Outline

- BFS Algorithm
- **BFS Application: Shortest Path on an unweighted graph**
- Unweighted Shortest Path: Proof of Correctness

Application: Shortest Paths on an Unweighted Graph

- Goal: To recover the shortest paths from a source node *s* to all other reachable nodes *v* in a graph.
  - The length of each path and the paths themselves are returned.
- Notes:
  - There are an exponential number of possible paths
  - Analogous to level order traversal for trees
  - This problem is harder for general graphs than trees because of cycles!



Breadth-First Search

Input: Graph $G = (V, E)$ (directed or undirected) and source vertex $s \in V$.

Output:

$d[v] =$ shortest path distance $\delta(s, v)$ from s to v, $\forall v \in V$.

$\pi[v] = u$ such that (u, v) is last edge on a shortest path from s to v.

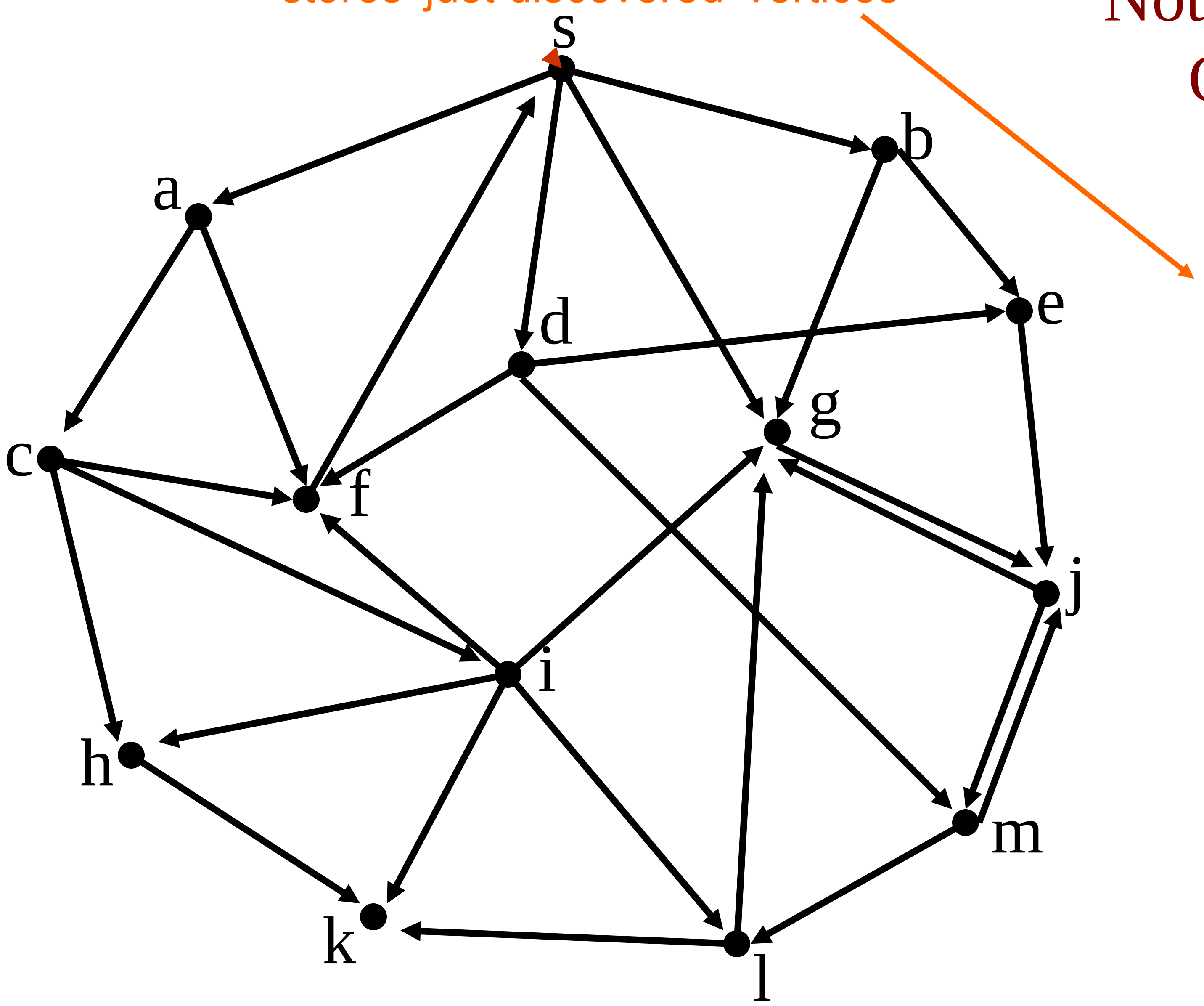
- Idea: send out search 'wave' from s.
- Keep track of progress by colouring vertices:
    - **Undiscovered** vertices are coloured **black**
    - **Just discovered** vertices (on the wavefront) are coloured **red.**
    - **Previously discovered** vertices (behind wavefront) are coloured **grey.**

BFS Algorithm with Distances and Predecessors

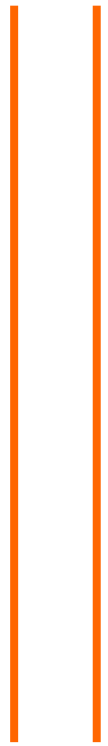
```
BFS(G,s)
Precondition: G is a graph, s is a vertex in G
Postcondition: d[u] = shortest distance d[u] and
p[u] = predecessor of u on shortest path from s to each vertex u in G
    for each vertex u Î V[G]
        d[u] ¬ ¥
        p[u] ¬ null
        color[u] = BLACK //initialize vertex
    colour[s] ¬ RED
    d[s] ¬ 0
    Q.enqueue(s)
    while Q ¹ Æ
        u ¬ Q.dequeue()
        for each v Î Adj[u] //explore edge (u,v)
            if color[v] = BLACK
                colour[v] ¬ RED
                d[v] ¬ d[u]+1
                p[v] ¬ u
                Q.enqueue(v)
        colour[u] ¬ GRAY
```

BFS

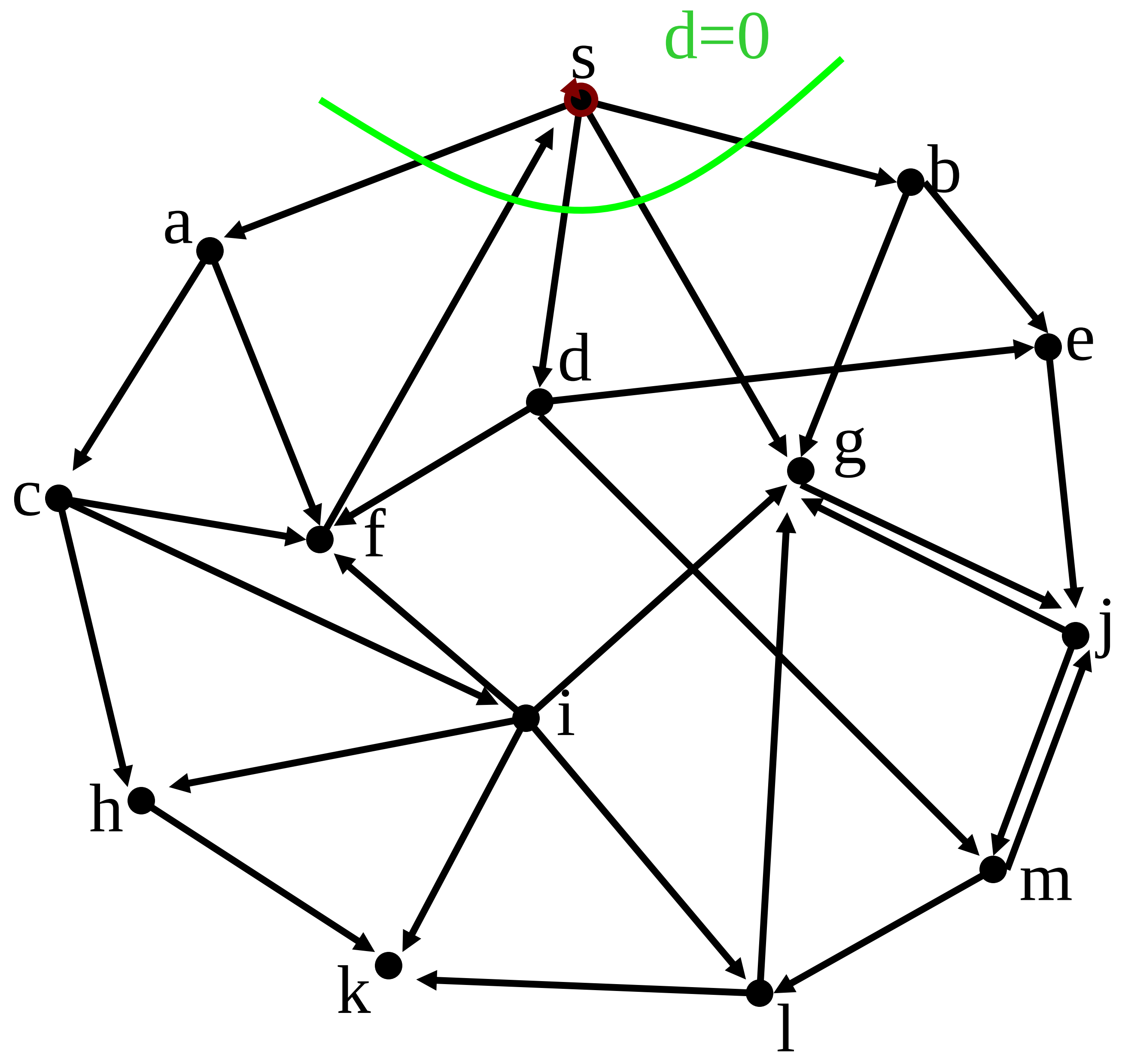
First-In First-Out (FIFO) queue
stores 'just discovered' vertices



Found
Not Handled
Queue



BFS



Found
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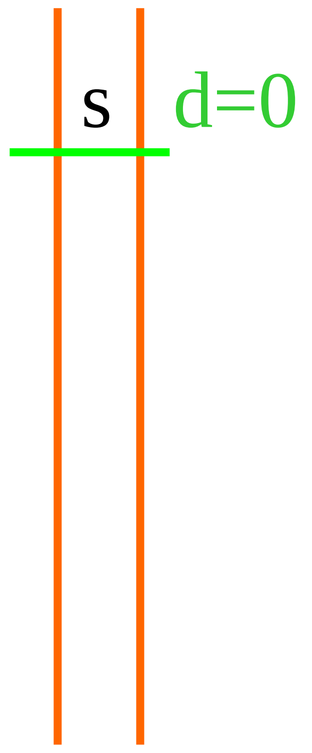
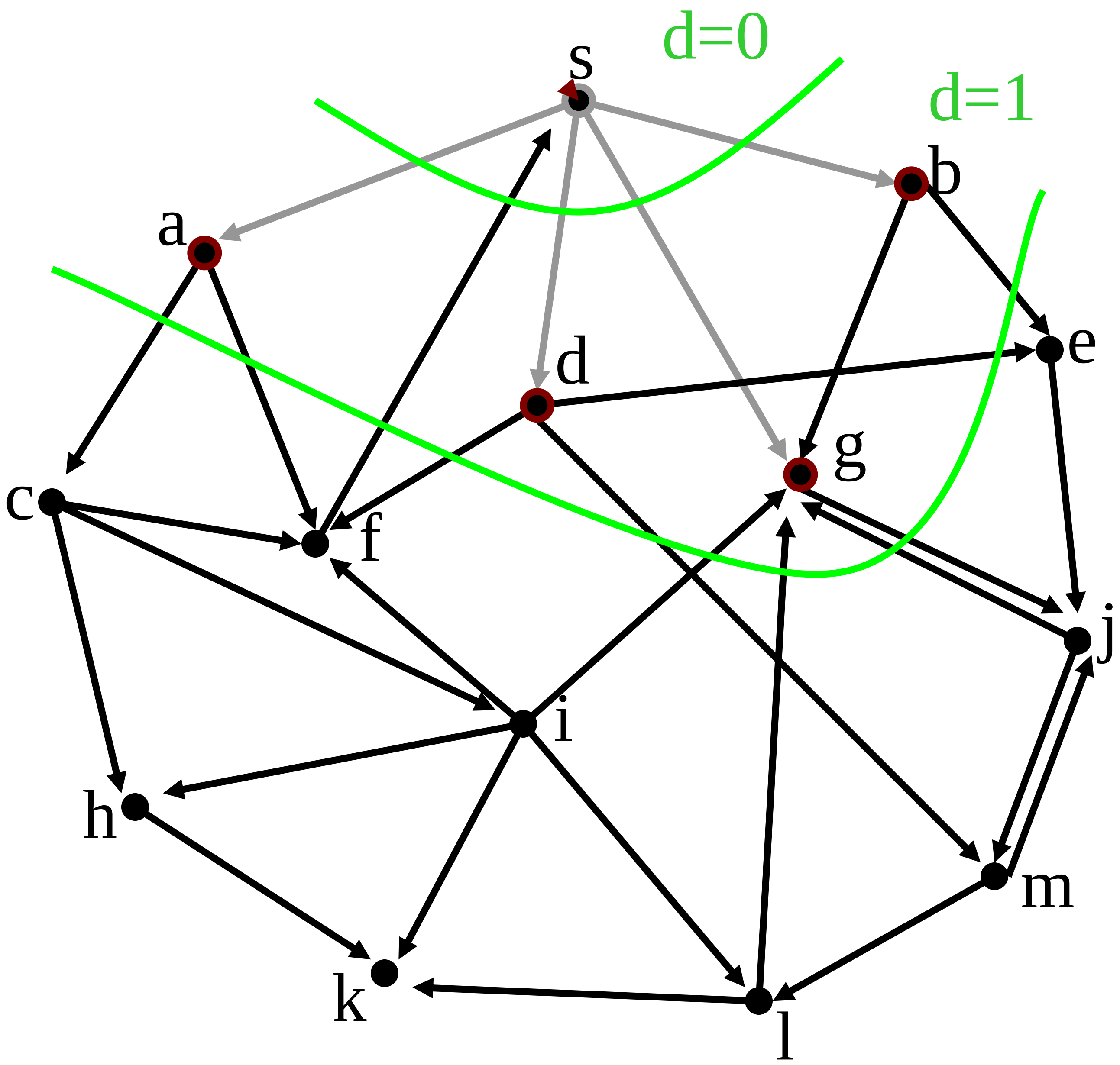
Queue



BFS



Found
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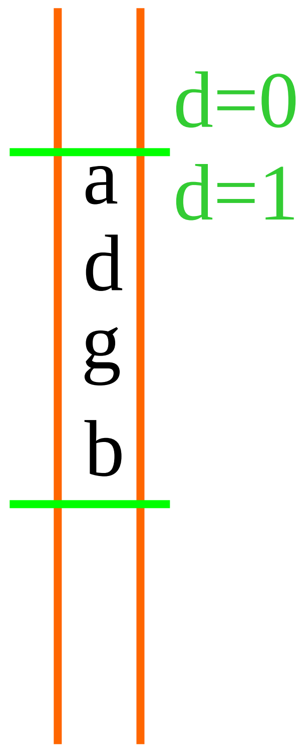
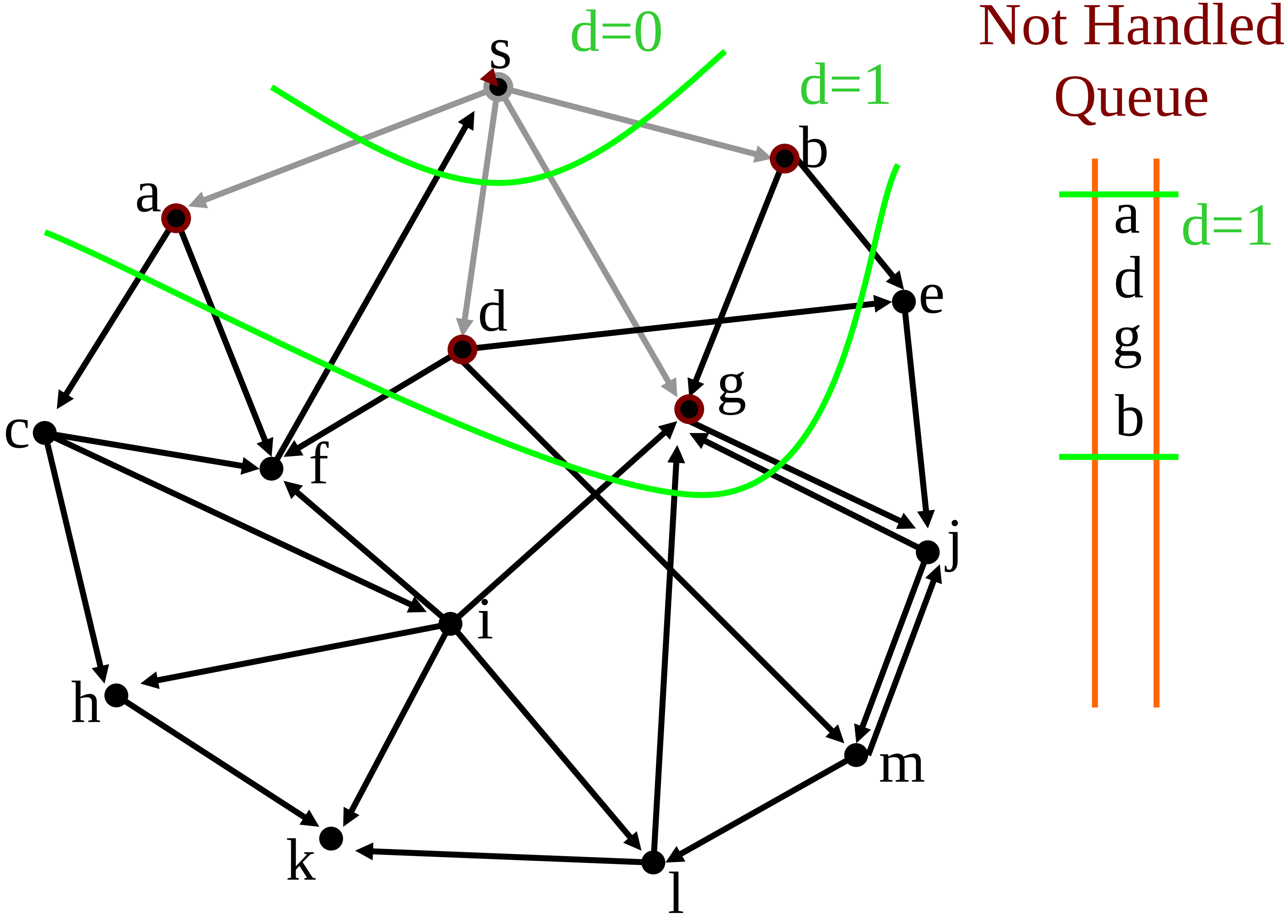
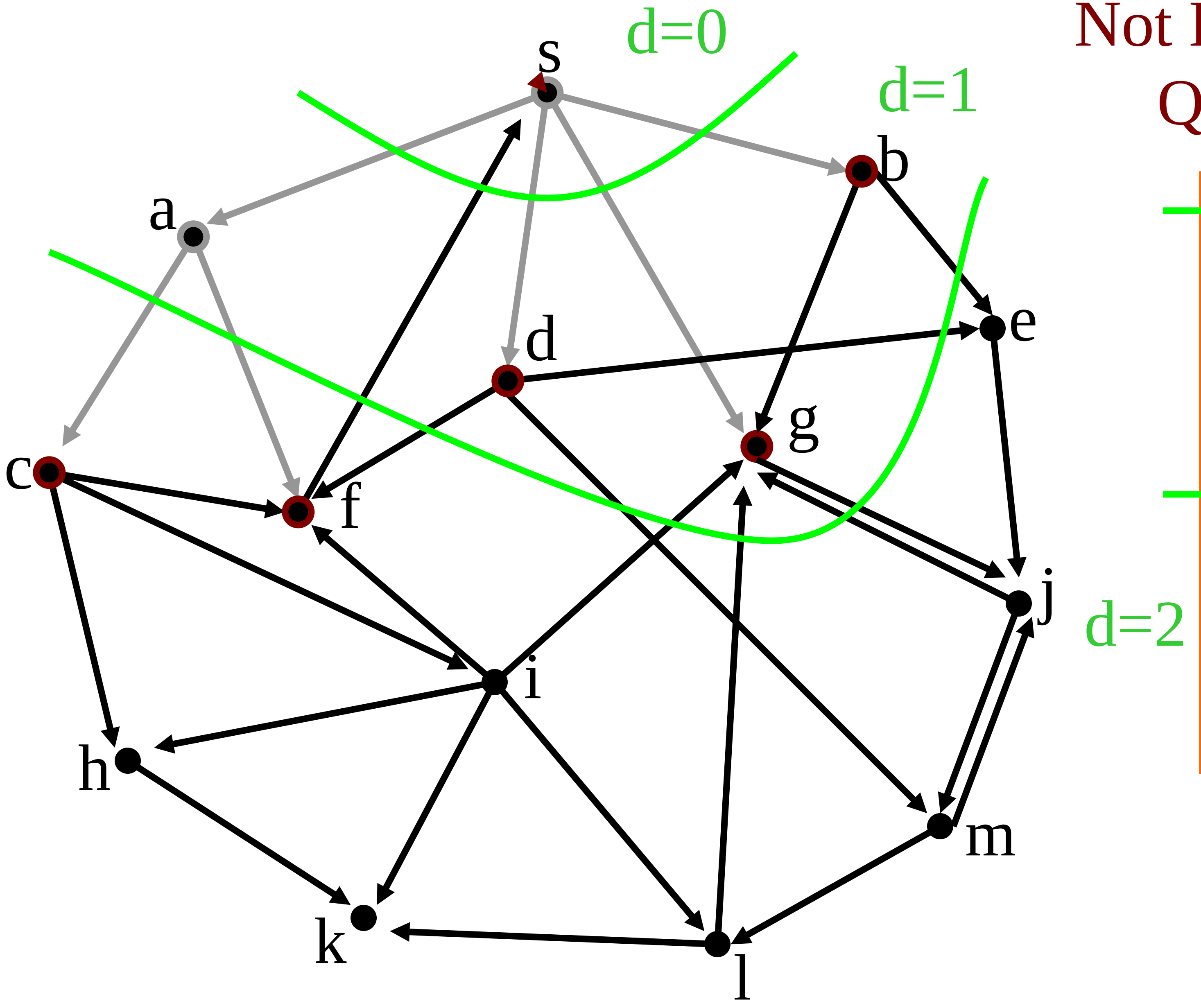
Queue



BFS



BFS



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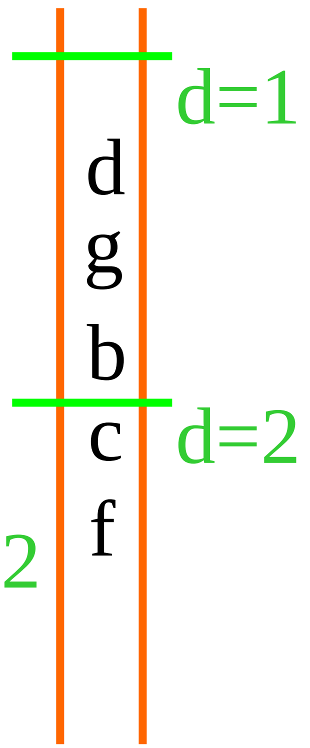
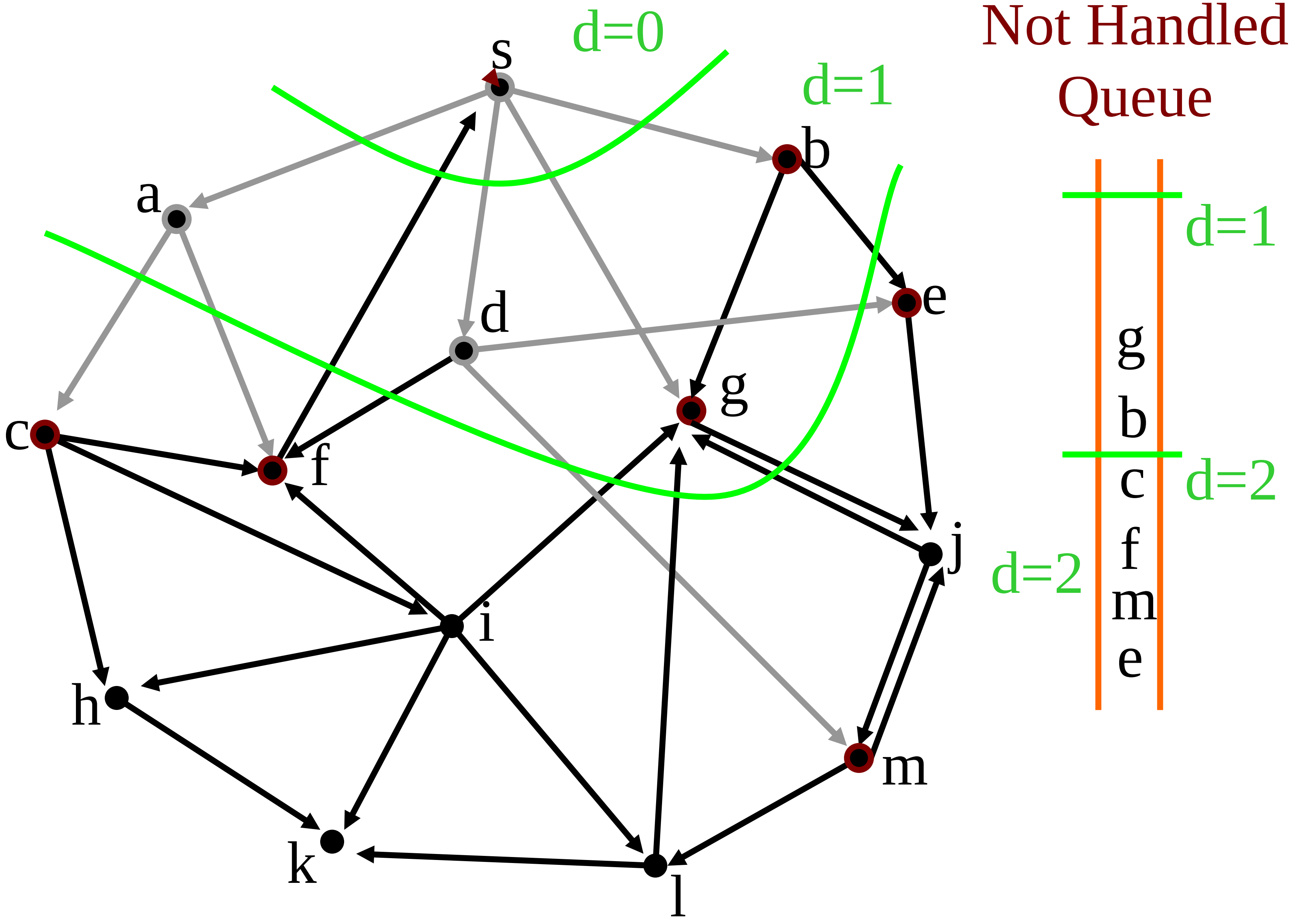


BFS



d=0
d=1
d=2
s
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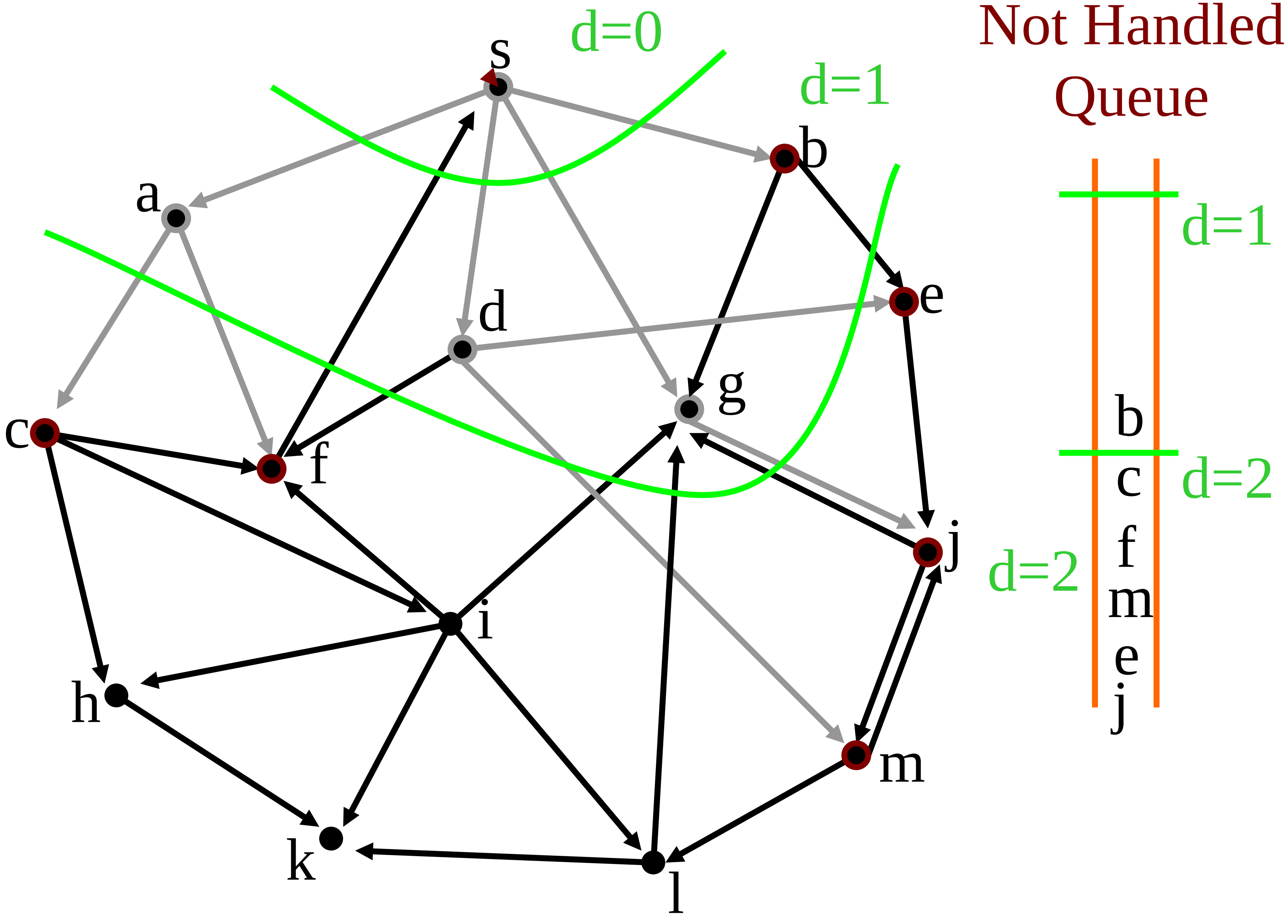
m
Found
Not Handled
Queue
d=1
g
b
d=2
c
f
m
e

BFS


Found
Not Handled
Queue
d=0
d=1
d=1
d=2
d=2
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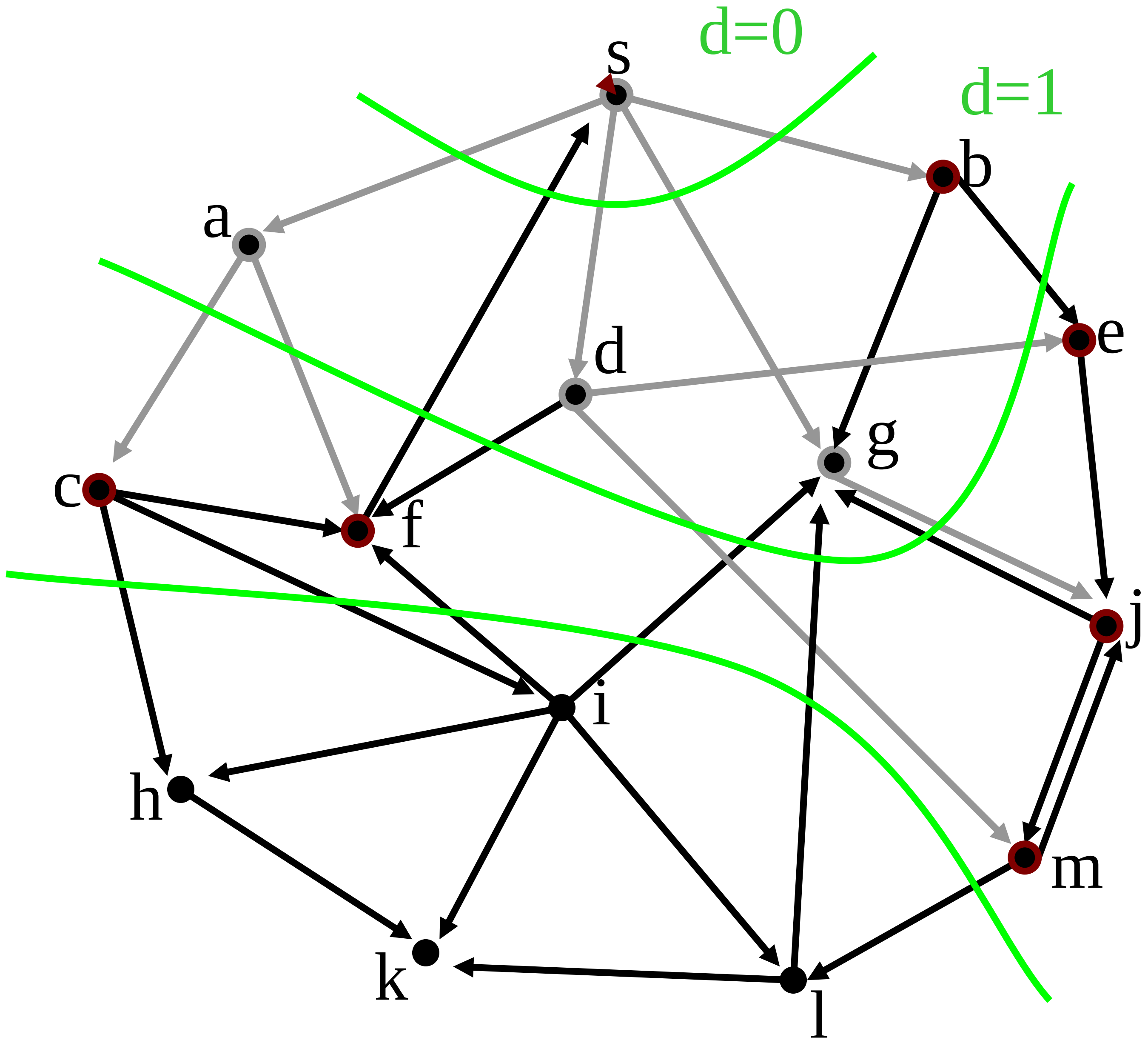
BFS



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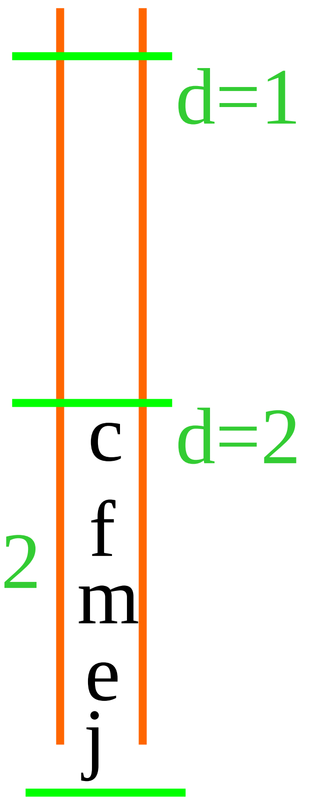
Queue



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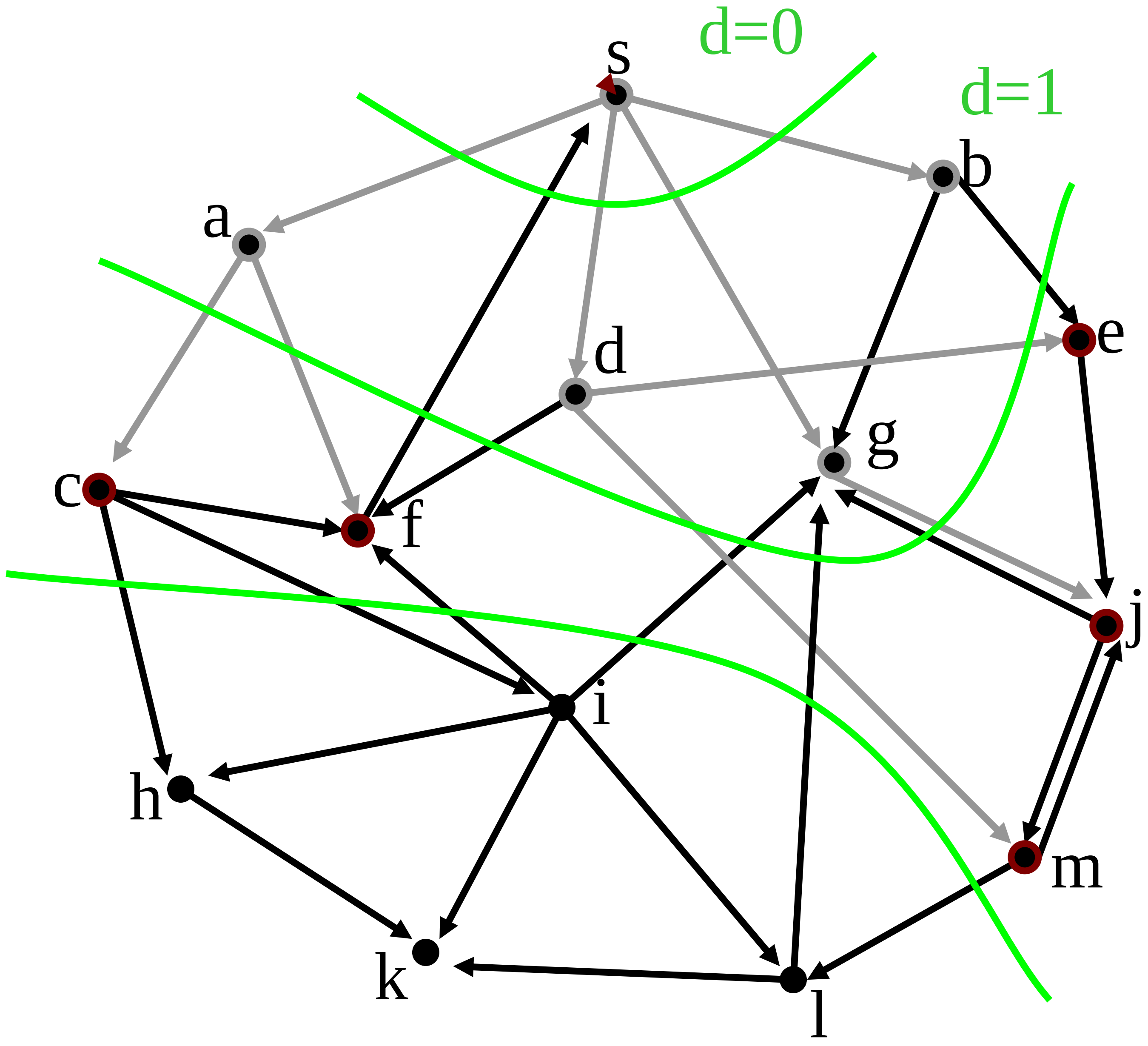

Found
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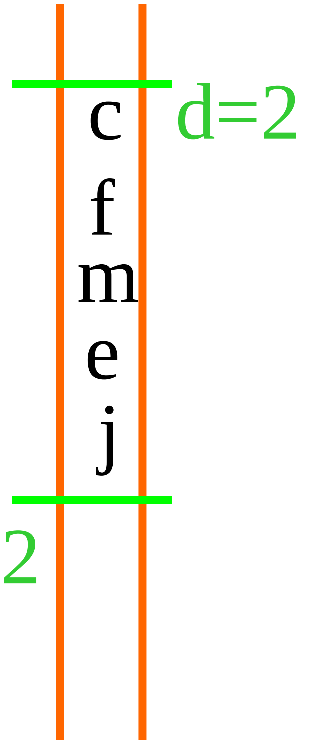
Queue



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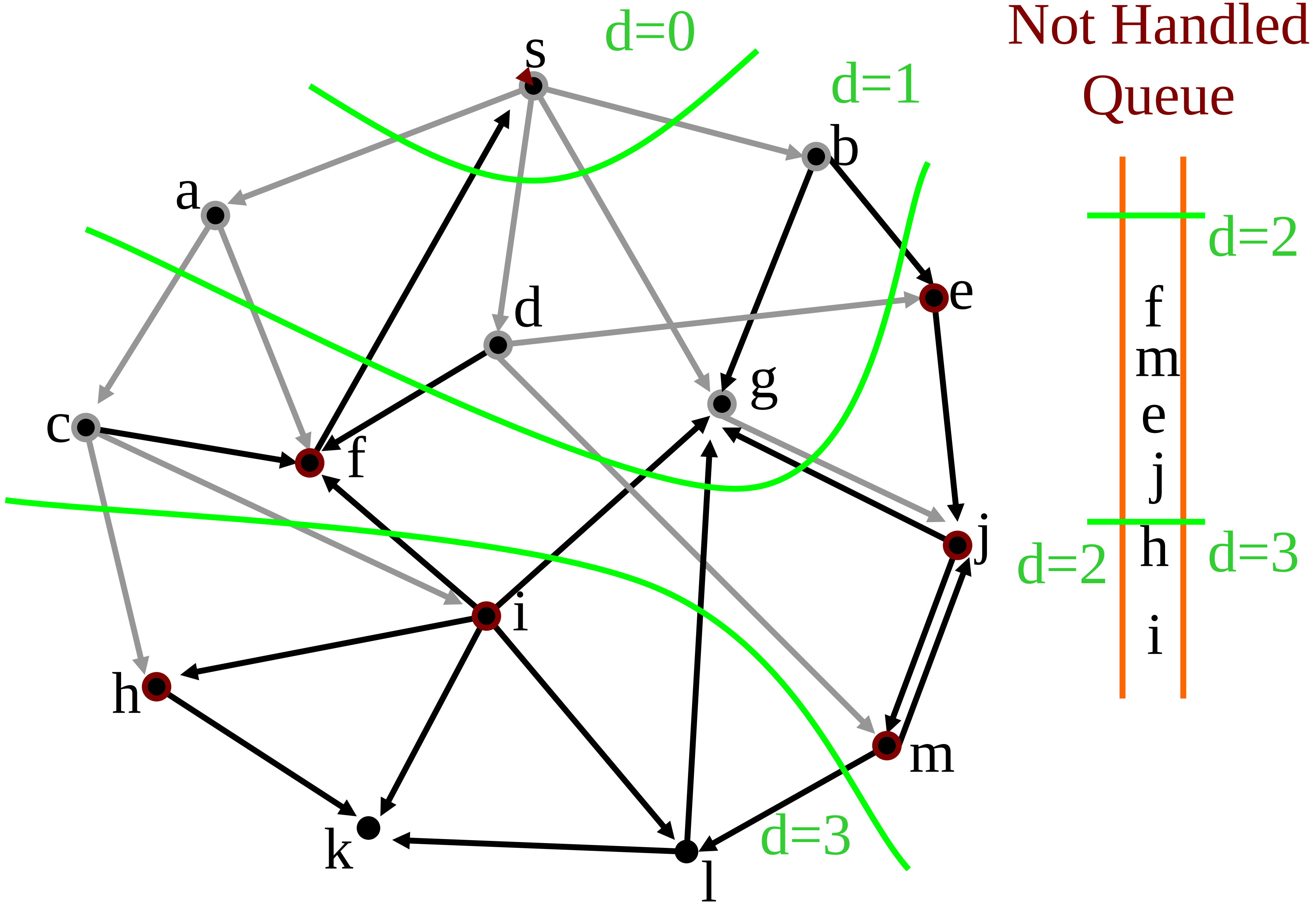
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d=3
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d
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f
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j
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l
m
Found
Not Handled
Queue
d=2
f
m
e
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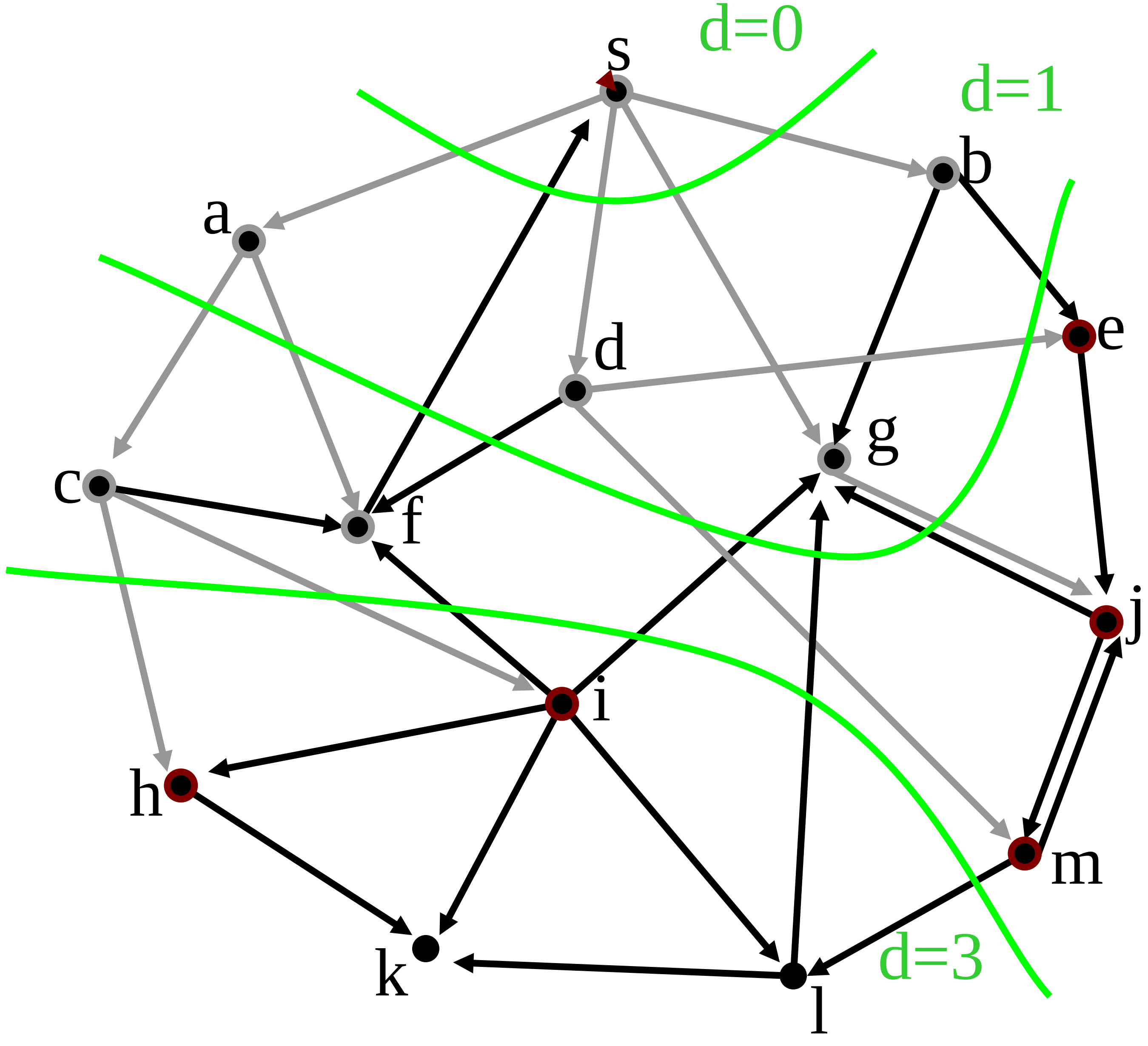
d=2
h
d=3
i

BFS



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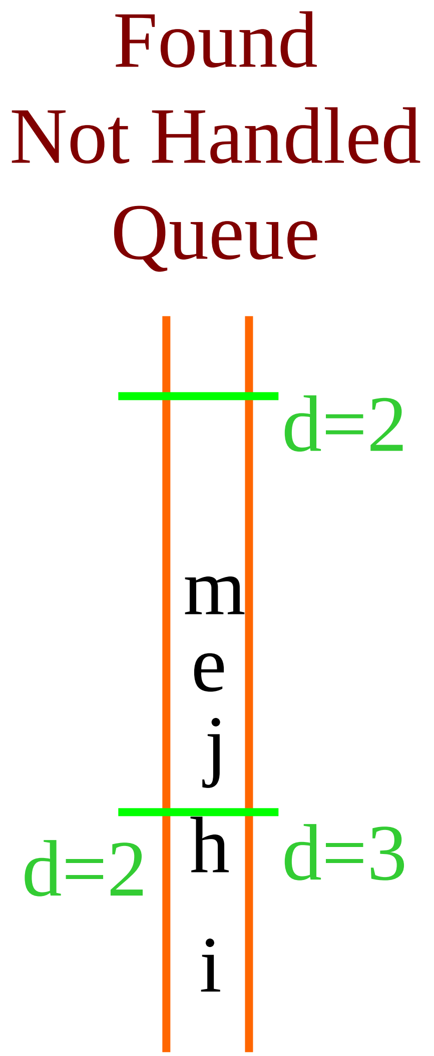
k
l
m
d=3



Found
Not Handled
Queue
d=2
m
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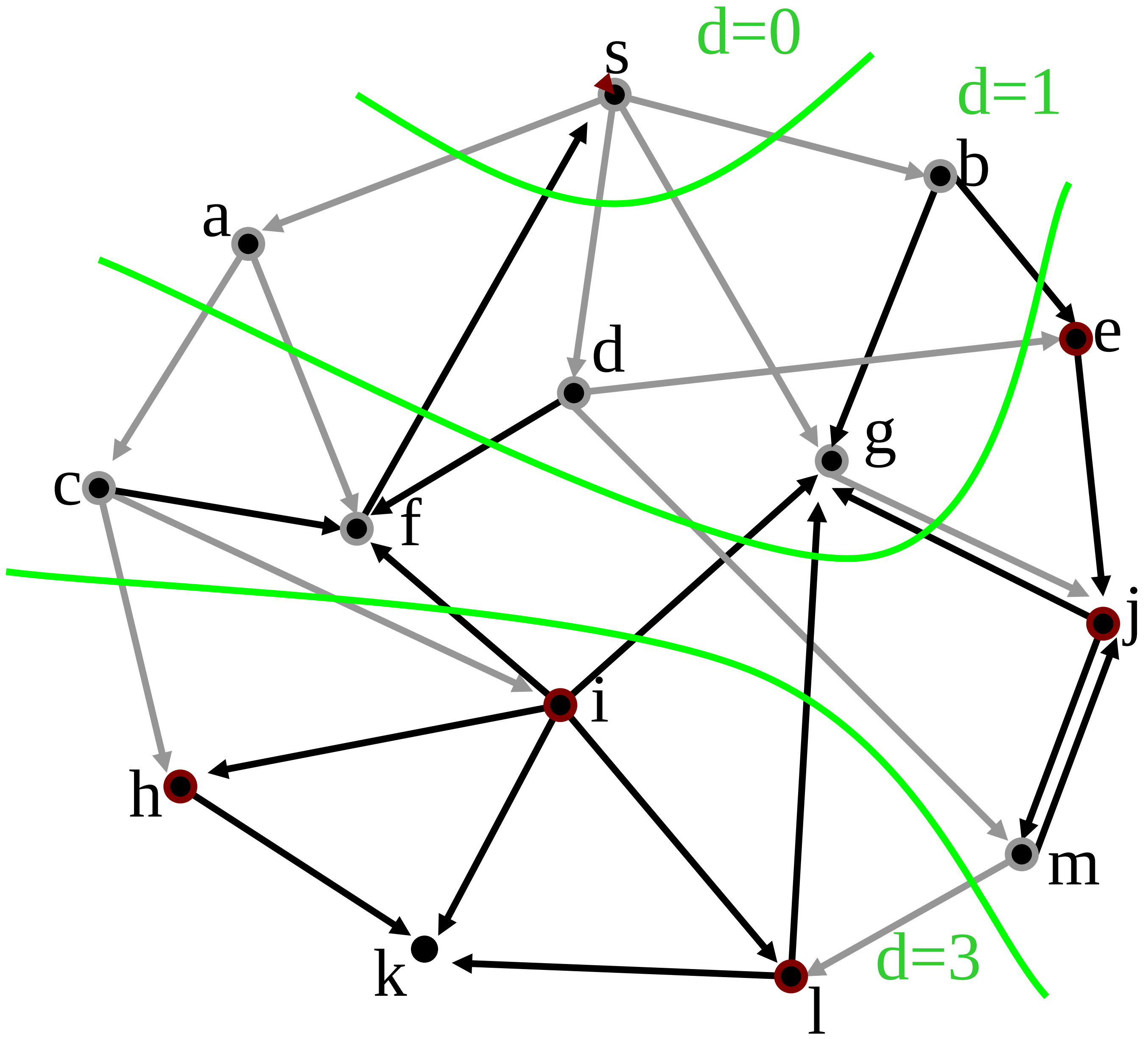
j
h
i
d=2
d=3

BFS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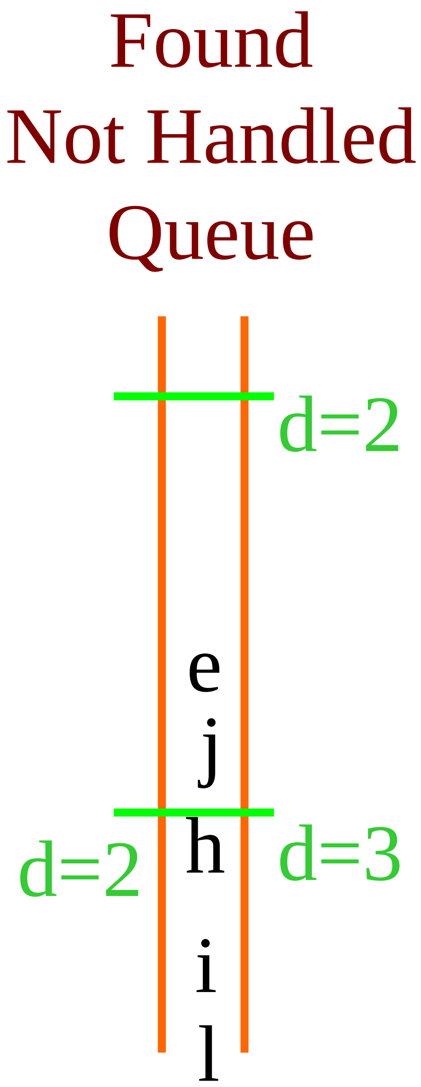
Found
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Queue
d=2
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d=2
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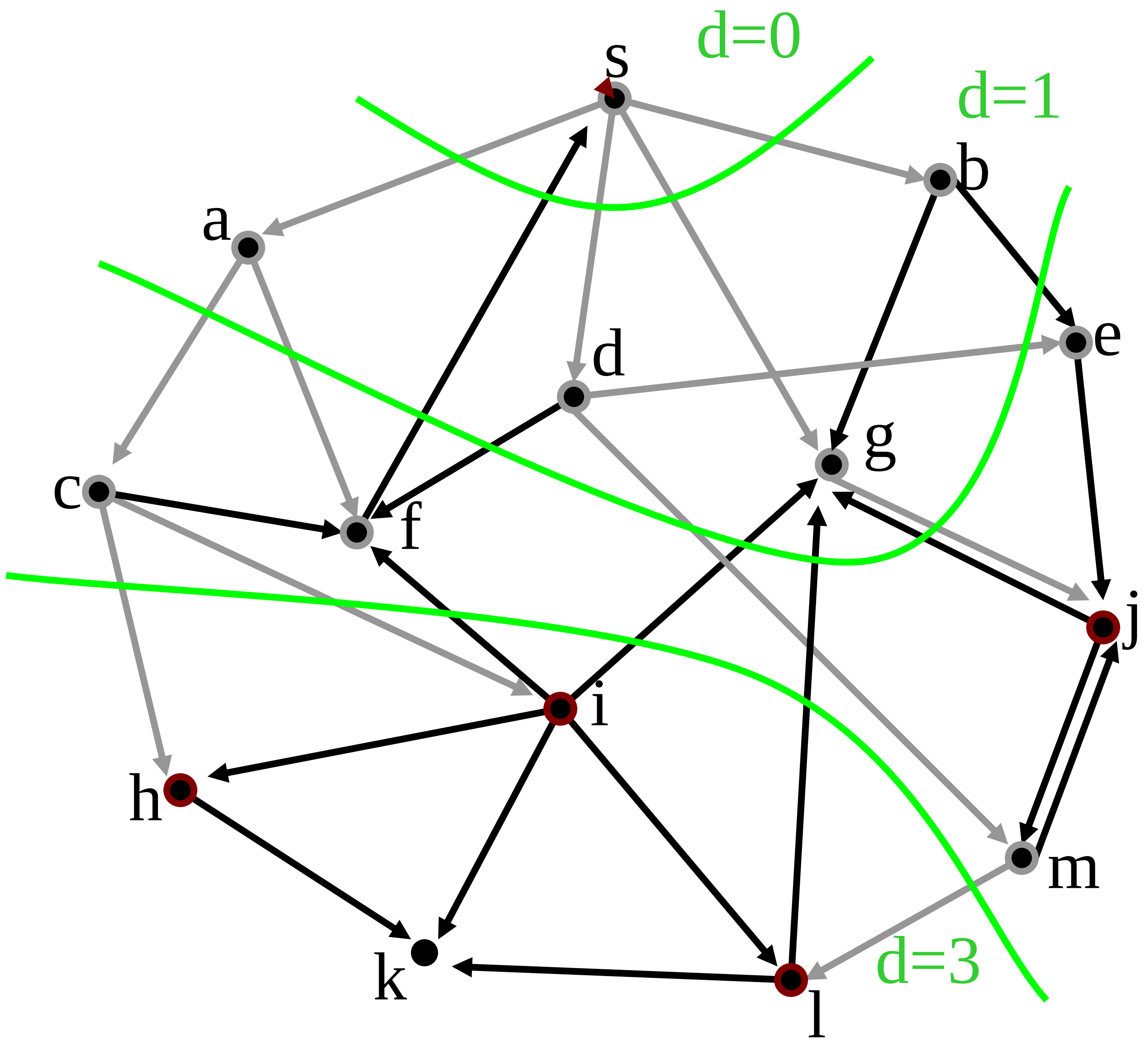
d=3
i
l

BFS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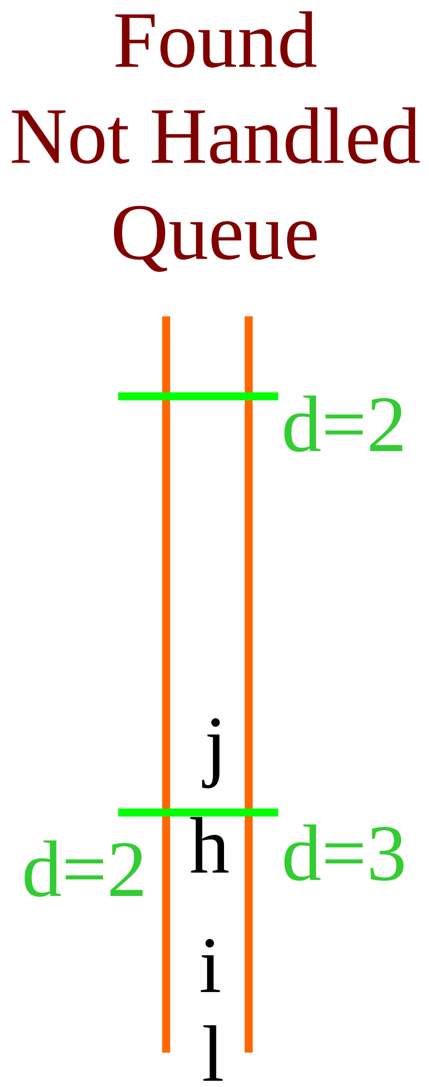
d=3


Found
Not Handled
Queue
d=2
j
d=2
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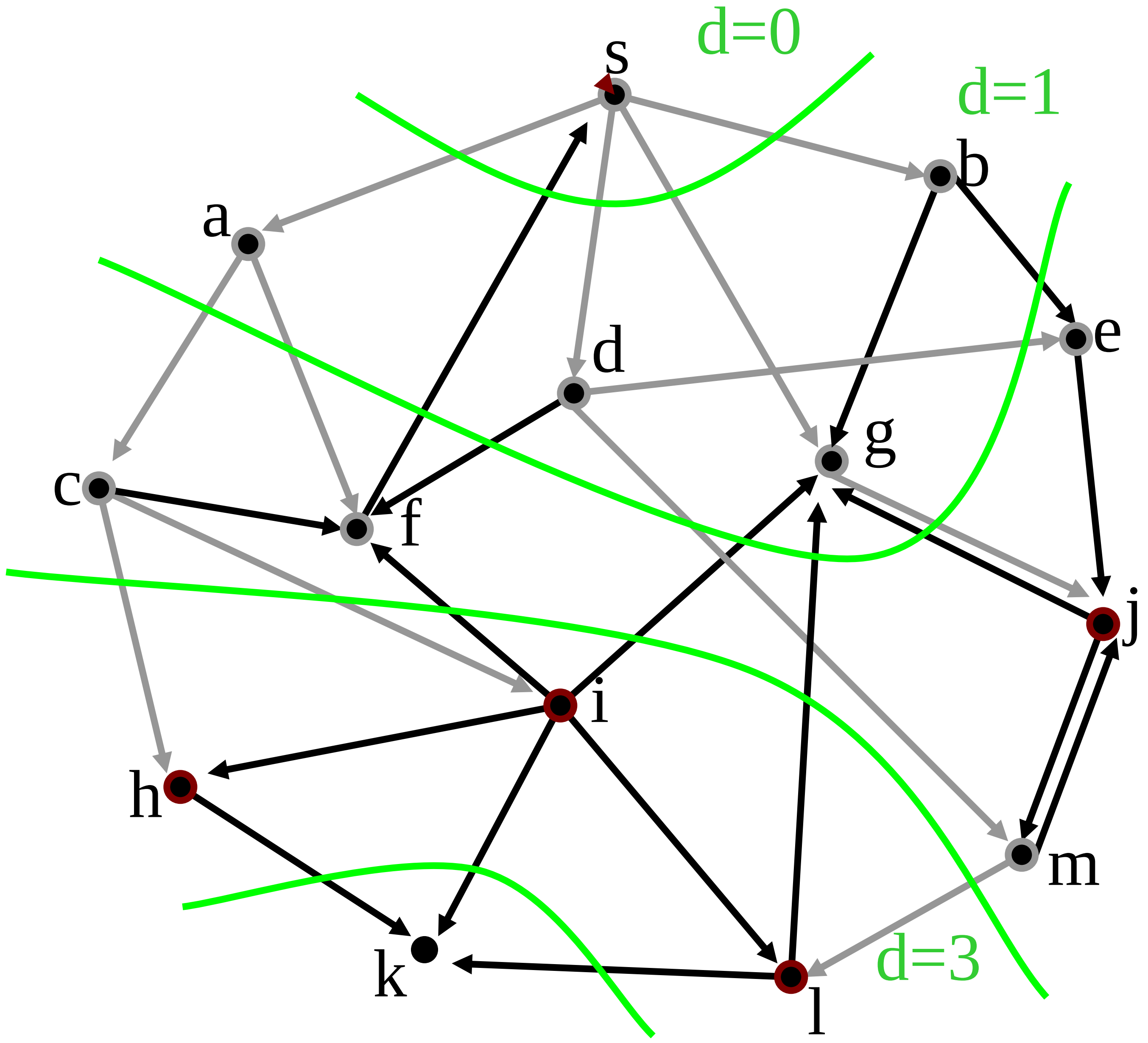
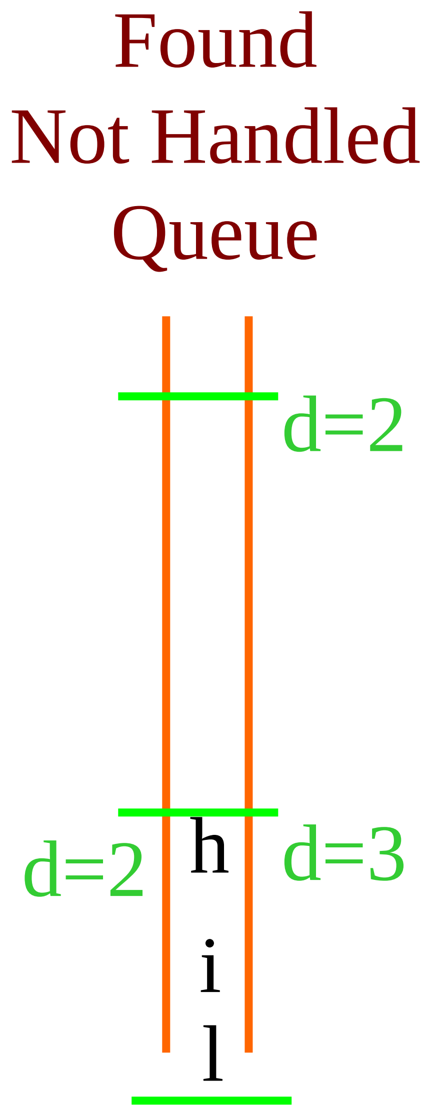
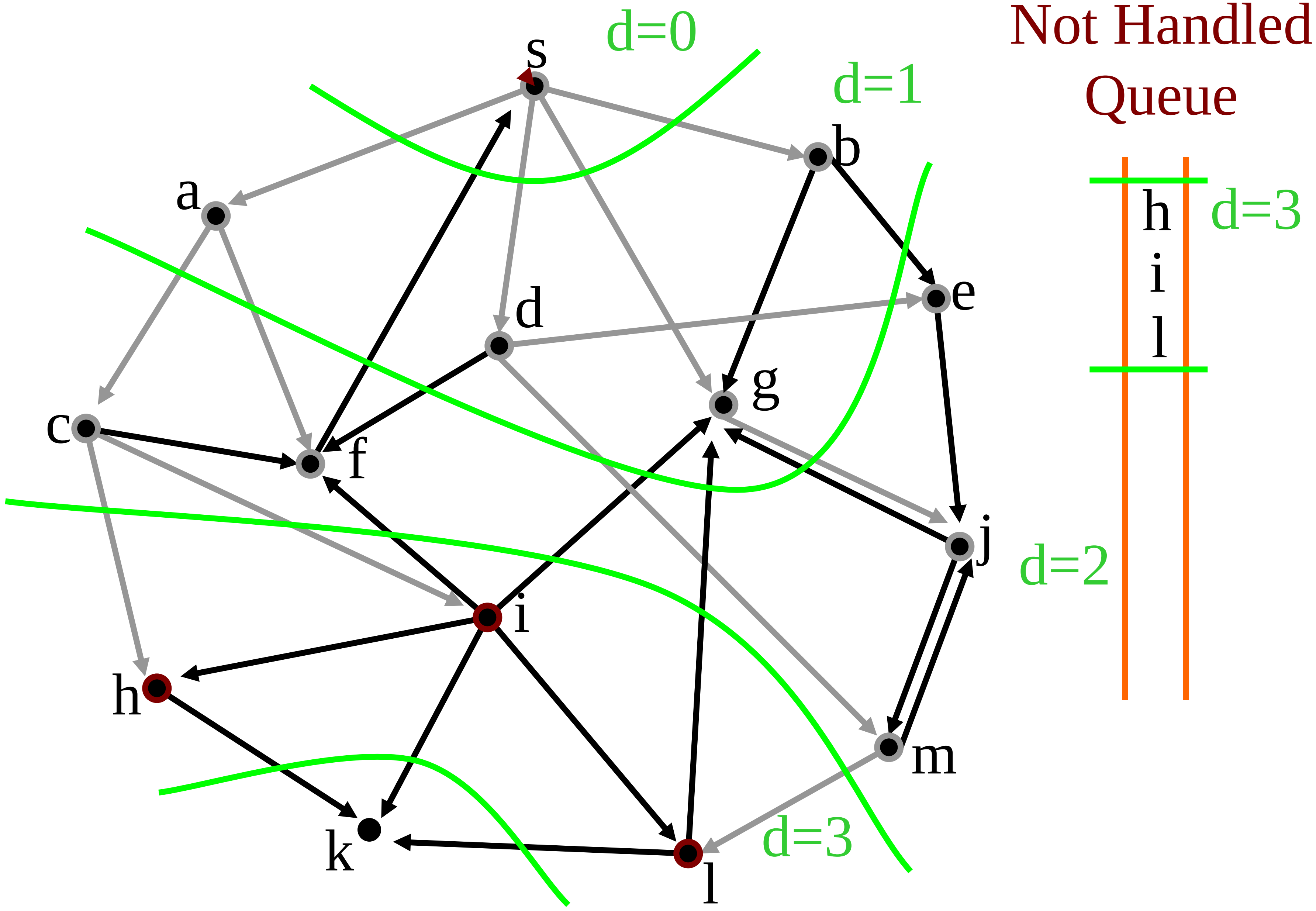
d=3
i
l

BFS





BFS



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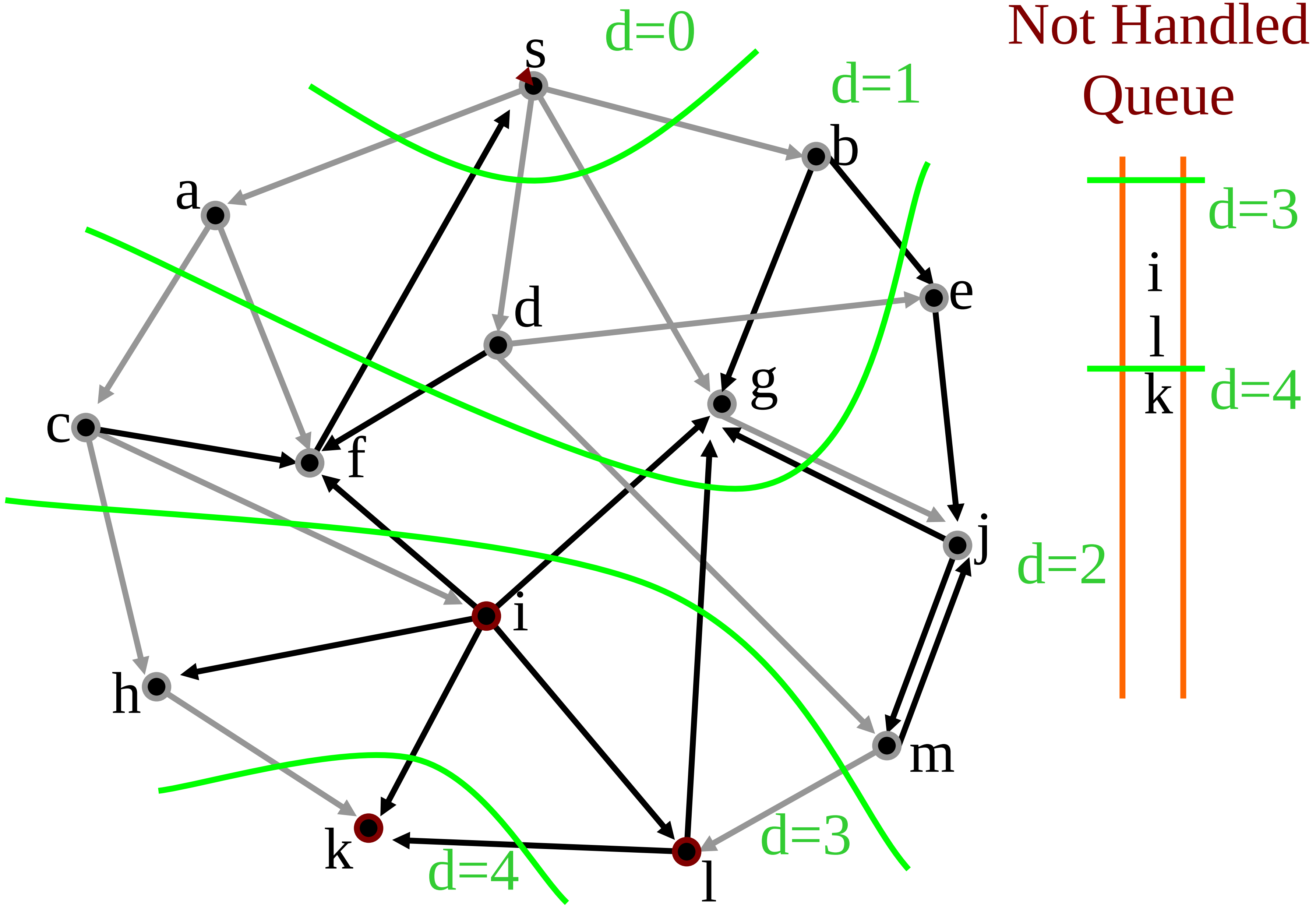
Found
Not Handled
Queue
d=0
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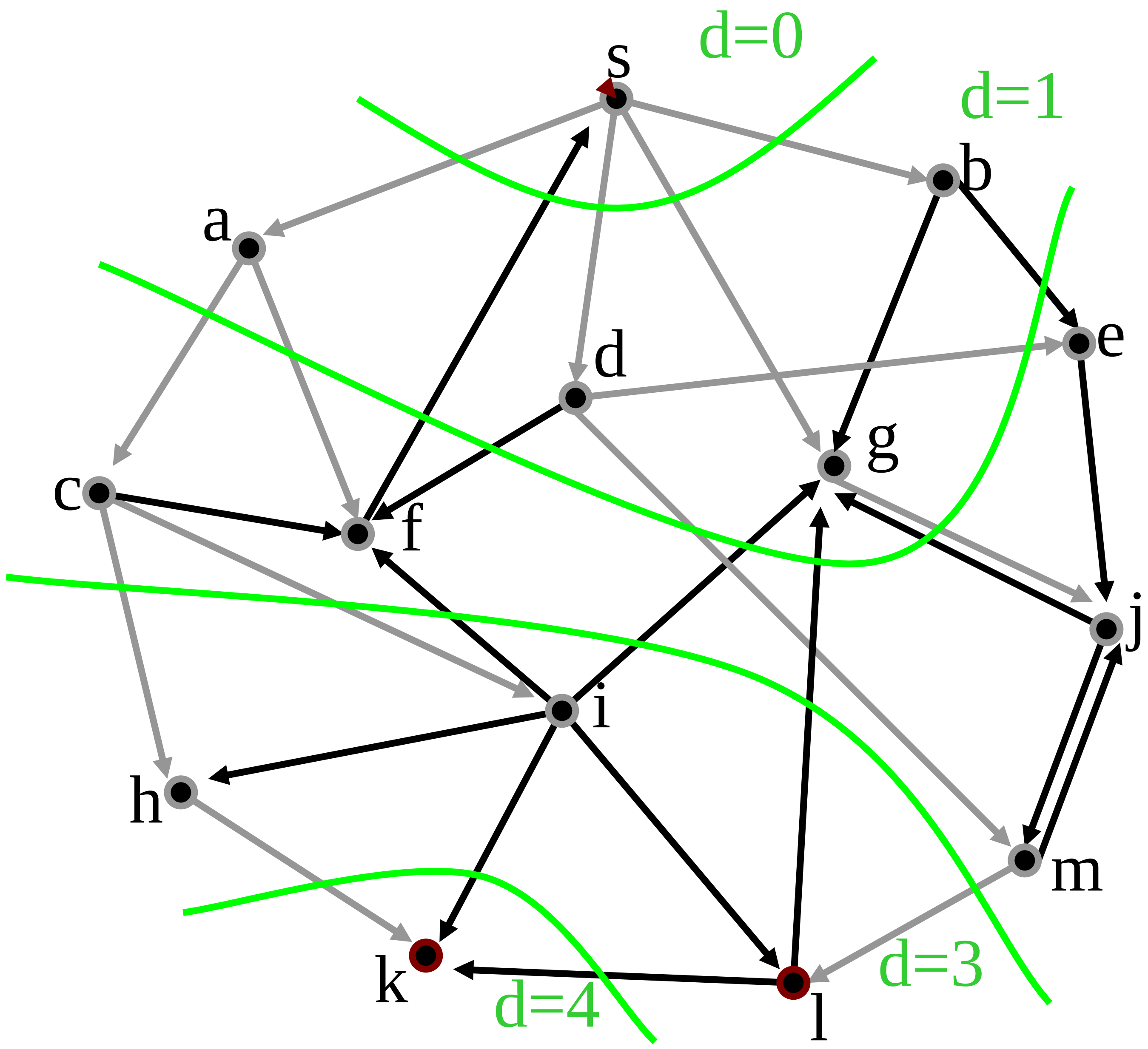
d=3
d=4

BFS



d=0
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d=4
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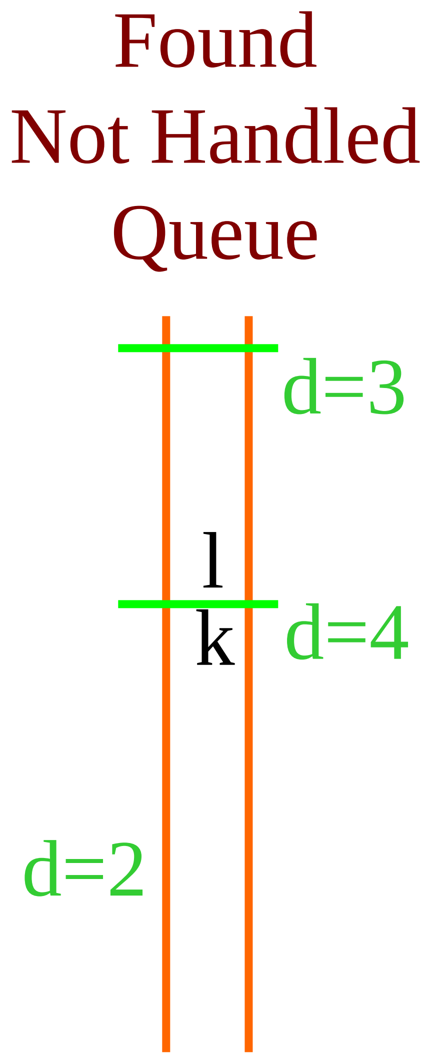
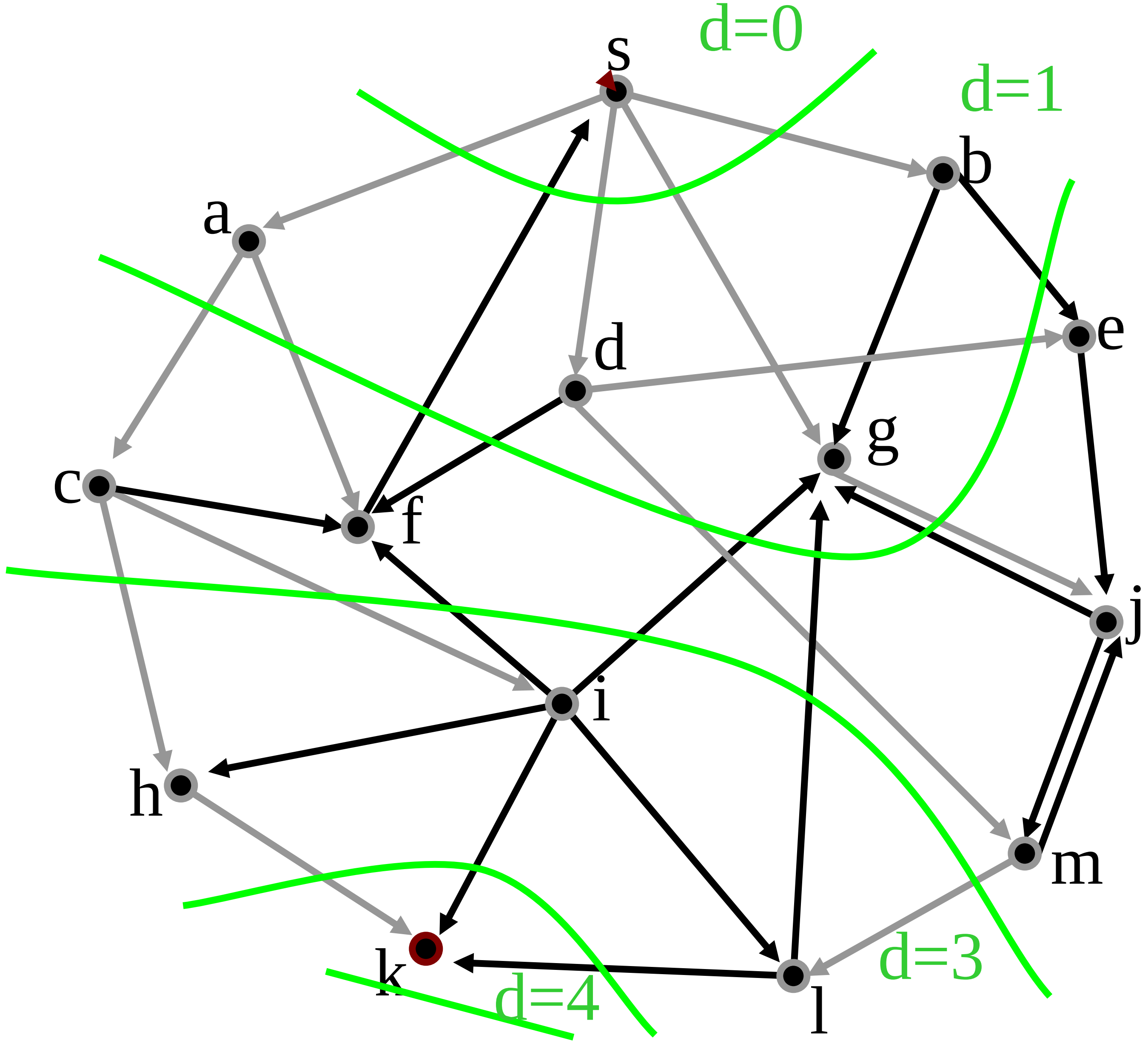
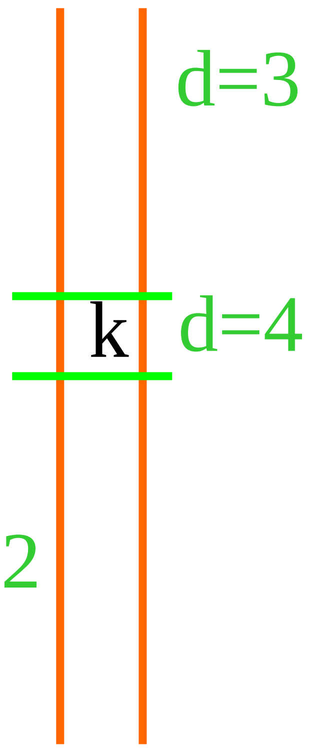


Found
Not Handled
Queue
d=3
l
k
d=4

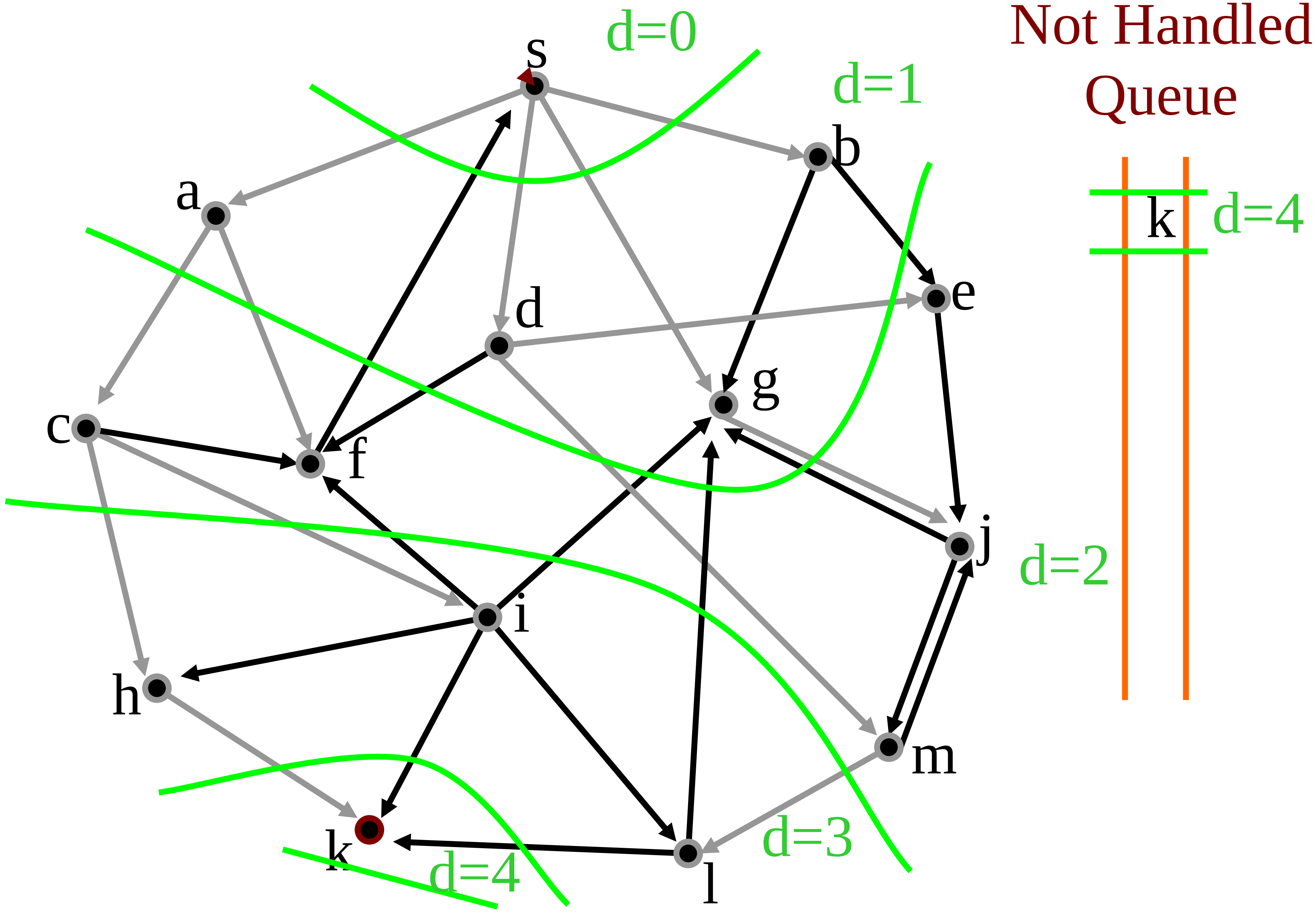
BFS



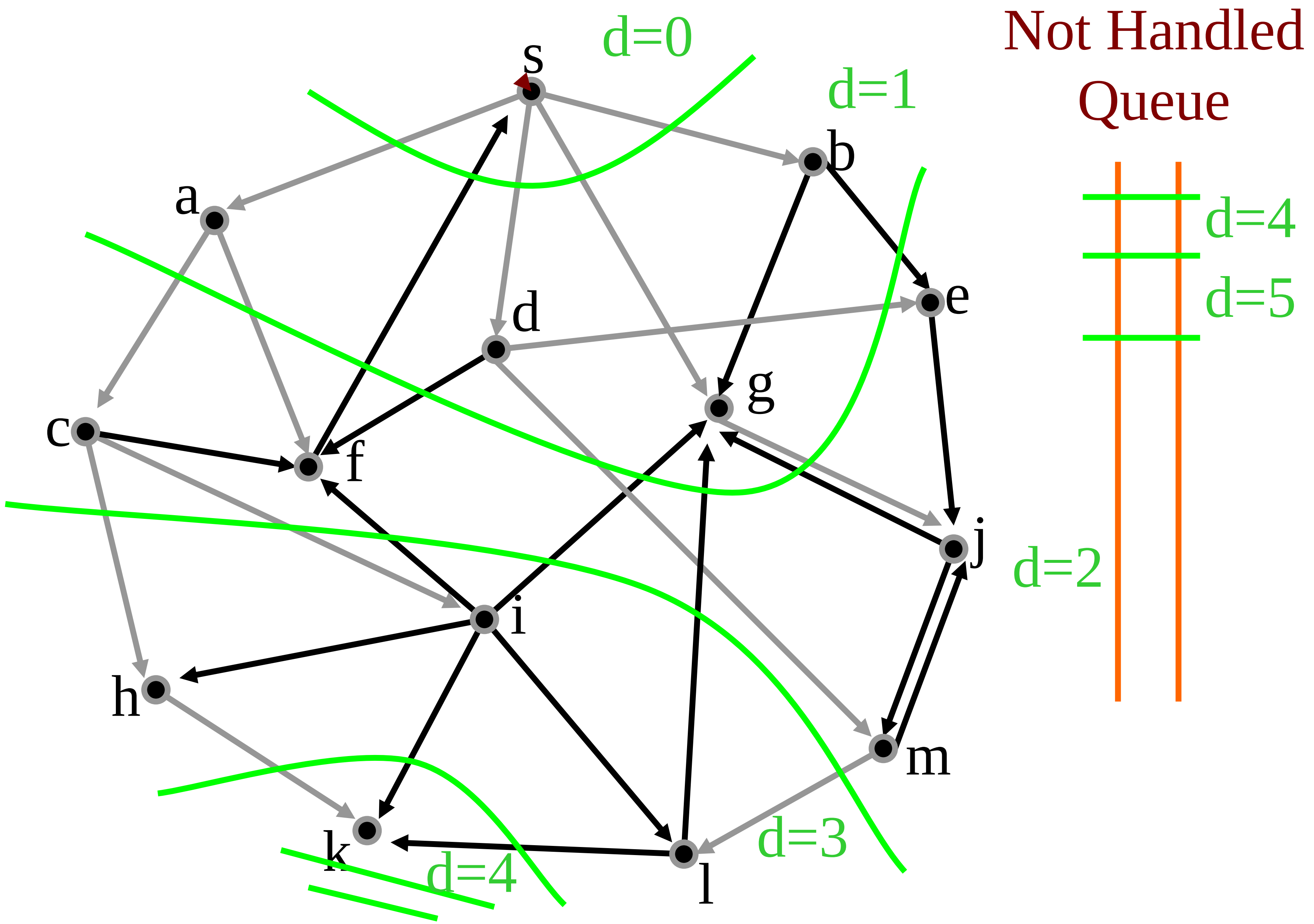
Found
Not Handled
Queue



BFS



BFS



Breadth-First Search Algorithm: Properties

```
BFS(G,s)
Precondition: G is a graph, s is a vertex in G
Postcondition: d[u] = shortest distance d[u] and
p[u] = predecessor of u on shortest paths from s to each vertex u in G
        for each vertex u Î V[G]
                d[u] ¬ ¥
                p[u] ¬ null
                color[u] = BLACK //initialize vertex
        colour[s] ¬ RED
        d[s] ¬ 0
        Q.enqueue(s)
        while Q ¹ Æ
                u ¬ Q.dequeue()
                for each v Î Adj[u] //explore edge (u,v)
                        if color[v] = BLACK
                                colour[v] ¬ RED
                                d[v] ¬ d[u] + 1
                                p[v] ¬ u
                                Q.enqueue(v)
                colour[u] ¬ GRAY
```

- Q is a FIFO queue.
- Each vertex assigned finite *d* value at most once.
- *Q* contains vertices with d values {*i*, …, *i*, *i*+*1*, …, *i*+*1*}
- *d* values assigned are monotonically increasing over time.

Breadth-First-Search is Greedy

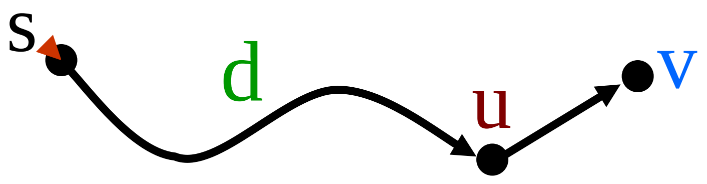
- Vertices are handled (and finished):
  - in order of their discovery (FIFO queue)
  - Smallest *d* values first

Outline

- BFS Algorithm
- BFS Application: Shortest Path on an unweighted graph
- **Unweighted Shortest Path: Proof of Correctness**

Correctness

Basic Steps:



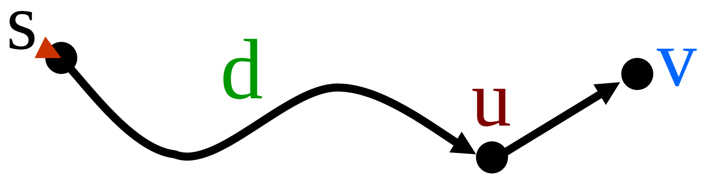
The shortest path to u has length d

& there is an edge from u to v

There is a path to v with length d+1.

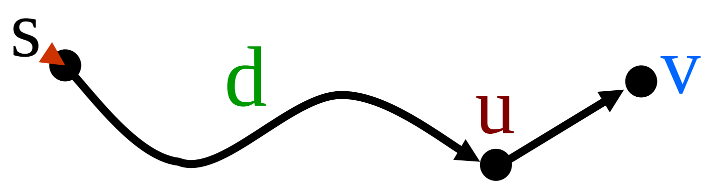
Correctness: Basic Intuition

- When we discover *v*, how do we know there is not a shorter path to *v*?
  - Because if there was, we would already have discovered it!



Correctness: More Complete Explanation

- Vertices are discovered in order of their distance from the source vertex *s*.
- Suppose that at time t_1 we have discovered the set V_d of all vertices that are a distance of *d* from *s*.
- Each vertex in the set V_{d+1} of all vertices a distance of *d*+1 from *s* must be adjacent to a vertex in V_d
- Thus we can correctly label these vertices by visiting all vertices in the adjacency lists of vertices in V_d.



Correctness: Formal Proof

Input: Graph $G = (V, E)$ (directed or undirected) and source vertex $s \in V$.

Output:

$d[v] =$ distance d(v) from s to v, " v Î V.

$p[v] = u$ such that (u,v) is last edge on shortest path from s to v.

Two-step proof:

On exit:

1. $d[v] \geq \delta(s,v) \forall v \in V$

2. $d[v] \ngtr \delta(s,v) \forall v \in V$

Claim 1. d is never too small: $d[v] \geq \delta(s,v) \forall v \in V$

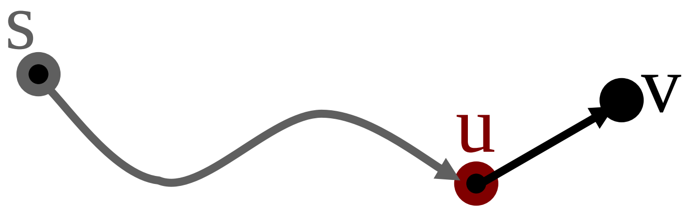
Proof: There exists a path from s to v of length £ $d[v]$.

By Induction:

Suppose it is true for all vertices thus far discovered (red and grey).

v is discovered from some adjacent vertex u being handled.

$$\begin{aligned} \rightarrow d[v] &= d[u]+1 \\ &\geq \delta(s,u)+1 \\ &\geq \delta(s,v) \end{aligned}$$



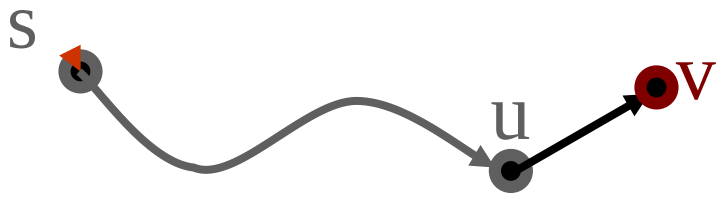
since each vertex v is assigned a d value exactly once,

it follows that on exit, $d[v] \geq \delta(s,v) \forall v \in V$.

Claim 1. *d* is never too small: $d[v] \geq \delta(s,v) \forall v \in V$

Proof: There exists a path from *s* to *v* of length £ *d*[*v*].

```
BFS(G,s)
Precondition: G is a graph, s is a vertex in G
Postcondition: d[u] = shortest distance d[u] and
p[u] = predecessor of u on shortest paths from s to each vertex u in G
    for each vertex u Î V[G]
        d[u] ¬ ¥
        p[u] ¬ null
        color[u] = BLACK //initialize vertex
    colour[s] ¬ RED
    d[s] ¬ 0
    Q.enqueue(s)
    while Q ¹ Æ      ← <LI>: $d[v] \geq \delta(s,v) \forall$ 'discovered' (red or grey) $v \in V$
        u ¬ Q.dequeue()
        for each v Î Adj[u] //explore edge (u,v)
            if color[v] = BLACK
                colour[v] ¬ RED
                d[v] ¬ d[u]+1      $\geq \delta(s,u)+1 \geq \delta(s,v)$
                p[v] ¬ u
                Q.enqueue(v)
        colour[u] ¬ GRAY
```



Claim 2. d is never too big: $d[v] \leq \delta(s,v) \forall v \in V$

Proof by contradiction:

Suppose one or more vertices receive a d value greater than δ.

Let v be the vertex with minimum $\delta(s,v)$ that receives such a d value.

Suppose that v is discovered and assigned this d value when vertex x is dequeued.

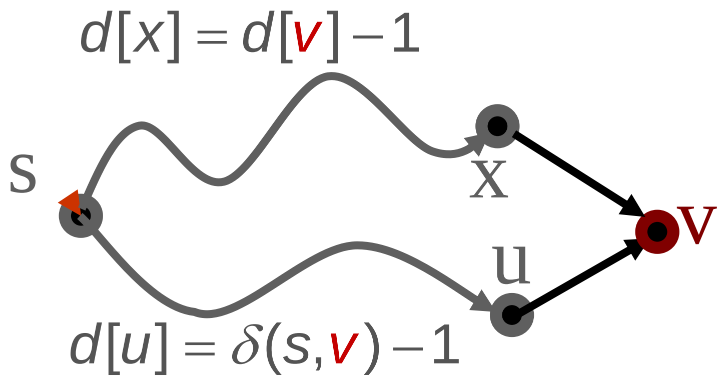
Let u be v's predecessor on a shortest path from s to v.

Then

$$\delta(s,v) < d[v]$$

$$\rightarrow \delta(s,v) - 1 < d[v] - 1$$

$$\rightarrow d[u] < d[x]$$



Recall: vertices are dequeued in increasing order of d value.

$\rightarrow$ u was dequeued before x.

$\rightarrow d[v] = d[u] + 1 = \delta(s,v)$ Contradiction!

Correctness

Claim 1. d is never too small: $d[v] \geq \delta(s,v) \forall v \in V$

Claim 2. d is never too big: $d[v] \leq \delta(s,v) \forall v \in V$

$\Rightarrow d$ is just right: $d[v] = \delta(s,v) \forall v \in V$

Progress?

- On every iteration one vertex is processed (turns gray).

```
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        Q.enqueue(s)
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                u ¬ Q.dequeue()
                for each v Î Adj[u] //explore edge (u,v)
                        if color[v] = BLACK
                                colour[v] ¬ RED
                                d[v] ¬ d[u] + 1
                                p[v] ¬ u
                                Q.enqueue(v)
                colour[u] ¬ GRAY
```
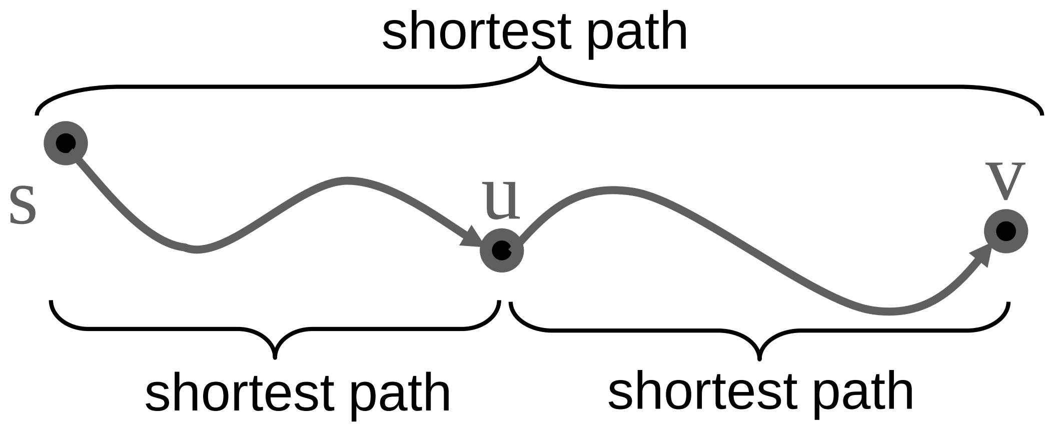

Optimal Substructure Property

- The shortest path problem has the optimal substructure property:
  - Every subpath of a shortest path is a shortest path.

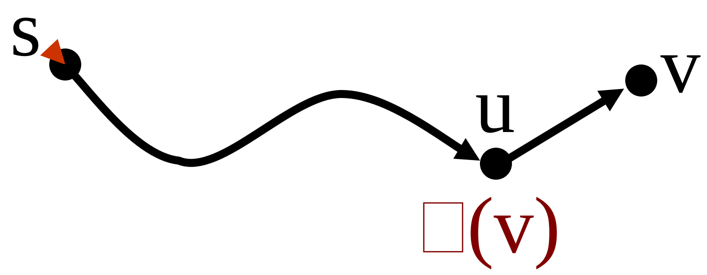
How would we prove this?

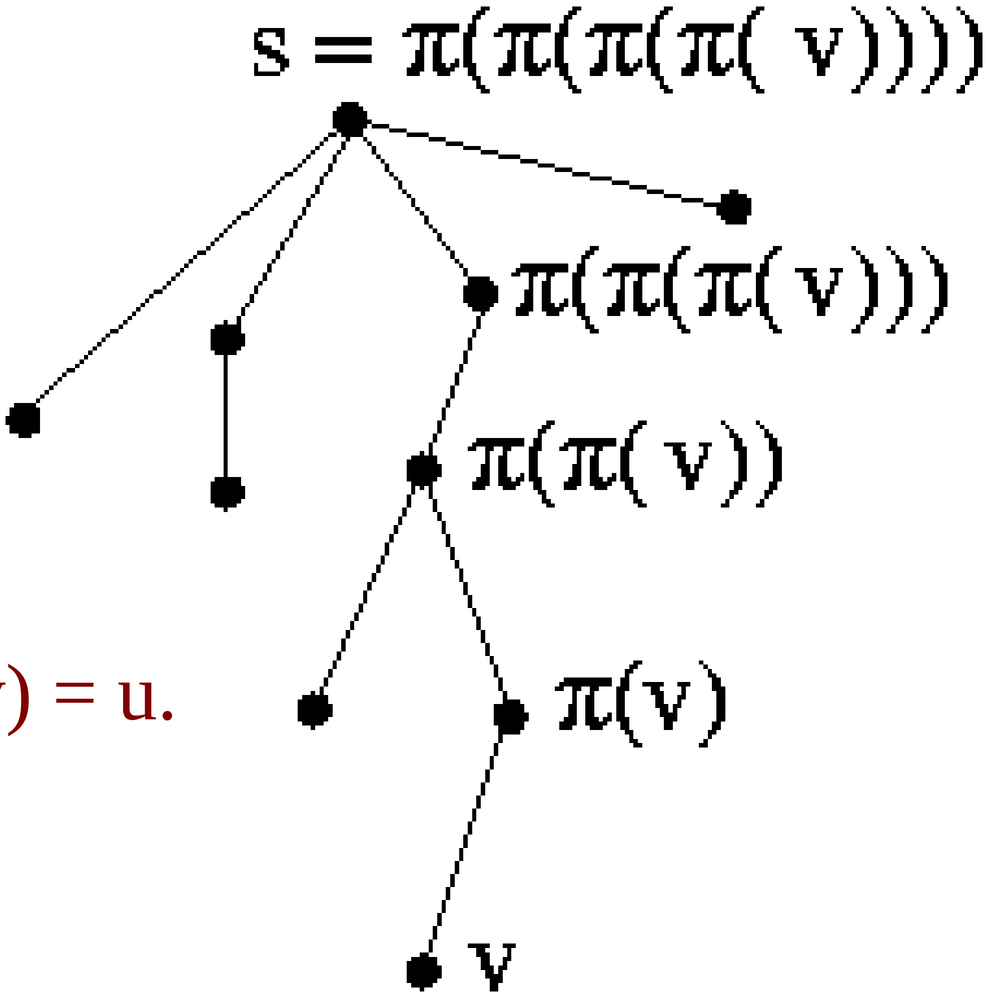


- The optimal substructure property
  - is a hallmark of both greedy and dynamic programming algorithms.
  - allows us to compute both shortest path distance and the shortest paths themselves by storing only one *d* value and one predecessor value per vertex.

Recovering the Shortest Path

For each node v, store predecessor of v in □(v).

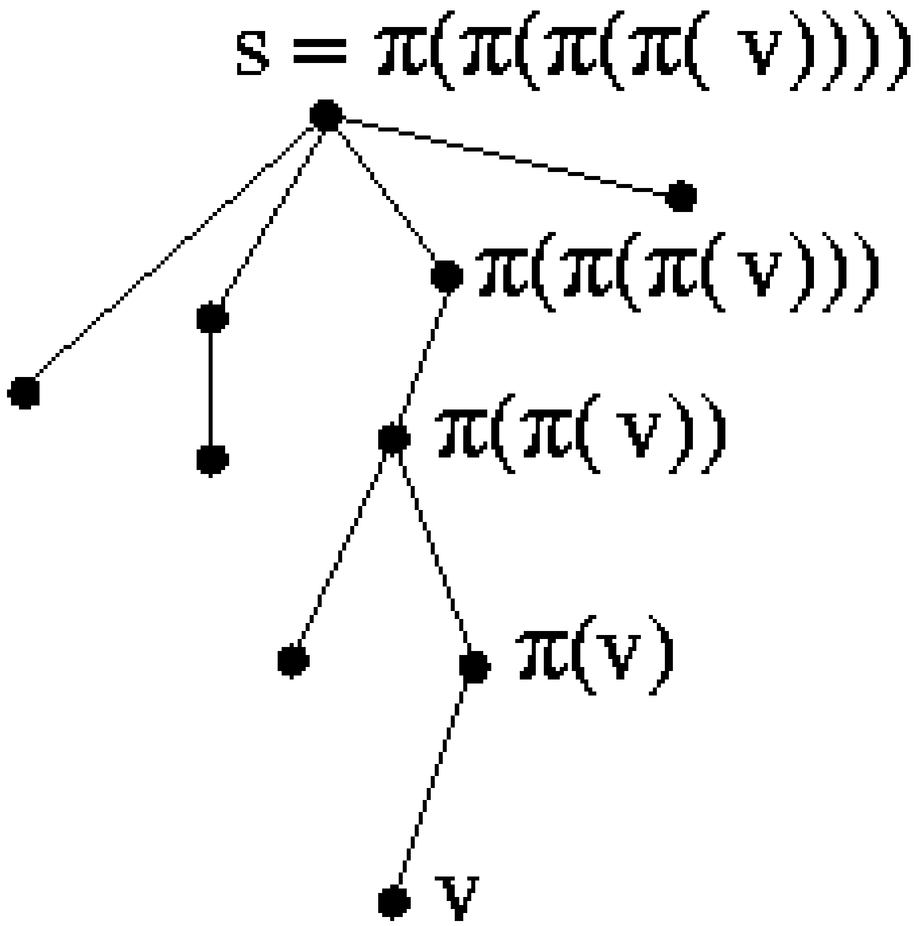




Predecessor of v is □(v) = u.

Recovering the Shortest Path

```
PRINT-PATH($G$, $s$, $v$)
Precondition: $s$ and $v$ are vertices of graph $G$
Postcondition: the vertices on the shortest path from $s$ to $v$ have been printed in order
if $v = s$ then
    print $s$
else if $\pi[v] = $ NIL then
    print "no path from" $s$ "to" $v$ "exists"
else
    PRINT-PATH($G$, $s$, $\pi[v]$)
    print $v$
```



BFS Algorithm without Colours

BFS(G,s)

Precondition: *G* is a graph, *s* is a vertex in *G*

Postcondition: predecessors *p*[u] and shortest distance *d*[*u*] from *s* to each vertex *u* in *G* has been computed

```
for each vertex u Î  V[G]
    d[u] ¬  ¥
    p[u] ¬  null
d[s] ¬  0
Q.enqueue(s)
while Q ¹  Æ
    u ¬  Q.dequeue()
    for each v Î  Adj[u] //explore edge (u,v)
        if d[v] = ¥
            d[v] ¬  d[u] + 1
            p[v] ¬  u
            Q.enqueue(v)
```

Outline

- BFS Algorithm
- BFS Application: Shortest Path on an unweighted graph
- Unweighted Shortest Path: Proof of Correctness